Perturbation growth in shear flow exhibits universality

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Disturbance structures that achieve maximum growth over a specified interval of time have recently been obtained for unbounded constant shear flow making use of closed-form solutions. Optimal perturbations have also been obtained for the canonical bounded shear flows, the Couette, and plane Poiseuille flows, using numerical solution of the linearized Navier–Stokes equations. In this note it is shown that these optimal perturbations have similar spectra and structure indicating an underlying universality of shear flow dynamics that is not revealed by traditional methods based on modal analysis.

It has been recognized since Reynolds[1] original investigation of transition to turbulence that shear flows at high Reynolds number are extremely sensitive to free-stream disturbances. This sensitivity is found even for very small perturbations for which the linearized Navier–Stokes equations are expected to accurately describe disturbance dynamics. Moreover, the canonical shear flow profiles, the Couette, plane Poiseuille, Hagen–Poiseuille, and unbounded constant shear are known to be asymptotically stable at Reynolds numbers for which transition is observed [of these only plane Poiseuille flow supports a modal instability and that at Reynolds numbers above 5772.22 (Ref. 2) while transition occurs for Reynolds numbers near 1000 (Ref. 3)]. Recently, it has been shown that a subset of perturbations in these shear flows exhibits very great transient growth and that this growth is associated with characteristic structures which are combinations of cross-stream waves and streamwise vortices.[4–6] The similarity of optimal growing structures in all these flows is to be contrasted with the dissimilarity of their modal spectra and associated normal modes and most particularly with the stability of these modal spectra.

A remarkable aspect of observations of shear flow turbulence is the universality of the structures obtained both by two-point correlation function analysis of laboratory experiments[7] and by visualization of numerical simulations.[8] A set of coherent structures is obtained by these techniques which appear to be characteristic of all such perturbations regardless of the precise nature of the background flow field. These structures do not correspond to the normal modes of the individual flows and their theoretical explanation has remained obscure. The structures referred to have been described variously as hairpin vortices, as roller eddies, and as streamwise vortices. Recent work on the optimal excitation of shear flow has demonstrated that the optimally growing perturbations reproduce salient features of these universal structures.[5,9]

In this Brief Communication the wave-number spectra of optimal growth and the associated optimal structures are compared for the examples of unbounded constant shear flow, Couette flow, and plane Poiseuille flow.

The linearized three-dimensional (3-D) Navier–Stokes equations governing evolution of disturbances in steady mean flow with x velocity $U(y)$ can be written in velocity/vorticity form as

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\Delta v-U_{yy}\frac{\partial}{\partial x}v-\frac{1}{R}\Delta\Delta v=0, \tag{1a}$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\omega-\frac{1}{R}\Delta\omega=-U_y\frac{\partial}{\partial z}v, \tag{1b}$$

where v is perturbation velocity in the y direction, ω is perturbation vorticity in the same direction, $\Delta\equiv\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$ is the Laplacian operator, and $R\equiv U_0L/\nu$ is the Reynolds number with U_0 the maximum velocity in a channel of half-width L, and ν the kinematic viscosity. We examine Couette flow for which $U=y$, and plane Poiseuille flow for which $U=1-y^2$. The boundary conditions imposed at $y=\pm 1$ in these bounded flows are $v=(\partial/\partial y)v=\omega=0$. For the unbounded constant shear flow we require boundedness at infinity. Because the unbounded flow has no intrinsic inertial scale, the Reynolds number in that case is defined as $R=S/\nu k^2$, where S is the shear and k a chosen wave-number scale for the perturbations.

Consider a single Fourier component:

$$v=\hat{v}(y)\exp(ikx+ilz), \tag{2a}$$

$$\omega=\hat{\omega}(y)\exp(ikx+ilz), \tag{2b}$$

in which physical variables are identified with the real part of these complex forms. The field equations (1) can be written in the compact form:

$$\frac{\partial}{\partial t}\begin{bmatrix}\hat{v}\\ \hat{\omega}\end{bmatrix}=\begin{bmatrix}\mathscr{L} & 0\\ \mathscr{C} & \mathscr{S}\end{bmatrix}\begin{bmatrix}\hat{v}\\ \hat{\omega}\end{bmatrix}, \tag{3}$$

in which the Orr–Sommerfeld operator $\mathscr{L}$, the Squire operator $\mathscr{S}$, and the coupling operator $\mathscr{C}$ are defined as

$$\mathscr{L}=\Delta^{-1}(-ikU\Delta+ikU_{yy}+\Delta\Delta/R), \tag{4a}$$

$$\mathscr{S}=-ikU+\Delta/R, \tag{4b}$$

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$$\mathscr{C} = -ilU_y, \tag{4c}$$

with $\alpha^2 \equiv k^2 + l^2$, and $\Delta \equiv (d^2/dy^2) - \alpha^2$.

A measure of perturbation growth is provided by the energy density associated with the Fourier component k, l:[10]

$$E_{k,l}(t) = \int \frac{1}{\alpha^2} \left(|\hat{\omega}|^2 + \alpha^2 |\hat{v}|^2 + \left| \frac{d}{dy} \hat{v} \right|^2 \right) dy, \tag{5}$$

with the appropriate limit for the infinite domain.

The maximal growth, $G_{k,l}$, of perturbations in the shear flow at wave numbers k, l and over a time interval T is given by

$$G_{k,l} = \mathrm{MAX}[E_{k,l}(T)/E_{k,l}(0)] \tag{6}$$

the maximization being over the initial cross-stream functional form of $\hat{v}(y)$ and $\hat{\omega}(y)$. For the unbounded constant shear flow the existence of closed-form solutions reduces the task of determining the maximal growth to a search over the initial configurations by means of a downhill simplex method.[5] Evolution of the initial perturbations for channel flows is determined from the discretized version of (3) and a variational procedure is used to determine the maximal growth.[4,11] There is some arbitrariness in the selection of T, but this is usefully resolved for turbulent flows as the time during which coherence of motion is maintained.[5,9] We choose to determine the maximal growth attained in ten advective time units ($T = 10$).

For the unbounded constant shear flow the maximal growth for $T = 10$ as a function of the wave number k, l [expressed in polar coordinates: $\alpha = (k^2 + l^2)^{1/2}$, $\Theta = \tan^{-1}(l/k)$] is shown in Fig. 1 for $R = 1000$. The Reynolds number is based on $\alpha = 1$. The axis $\Theta = 0°$ gives the growth of the 2-D cross-stream wave disturbances, while $\Theta = 90°$ is the axis of the streamwise rolls. Maximal growth is attained for $\Theta \approx 60°$. Note that the growth decreases rapidly for $\alpha > 1$, as the perturbations are subject to increased viscous dissipation. This viscous decay is forestalled along the roll axis indicating greater persistence of the streamwise vortices. All disturbances for $\Theta \neq 0°$ or $90°$, i.e., not pure cross-stream waves or pure rolls, demonstrate a universal mode of development in which at the time of maximal energy they assume a form much like that of Townsend's double rollers, shearing over with time to produce energetic streaks before dissipating.[5] For $\alpha \leqslant 0.4$ (corresponding to an equivalent Reynolds number $R \geqslant 6250$) the growth obtains a constant distribution which is characteristic of the inertial subrange in all shear flow examples. In the examples to follow we choose $R = 10\,000$ which places the corresponding Reynolds number within this inertial subrange.

The universal nature of the development process in shear flow is revealed by comparing the above with the maximal growth over the interval $T = 10$ for the Couette flow (Fig. 2) and plane Poiseuille flow (Fig. 3) at $R = 10\,000$. Note the remarkable similarity of the growth contours. The viscous decay for large α, the persistence of the roll solutions around $\Theta = 90°$, and the maximum for $\Theta \approx 60°$ have striking parallels in the three figures. The obvious difference is the decrease in growth for small α associated with the confinement of the perturbations arising from the imposition of boundaries in the Couette and plane Poiseuille flow. The similarity in the growth arises from similarity in the growth structures found to underlie the optimals.[4,5]

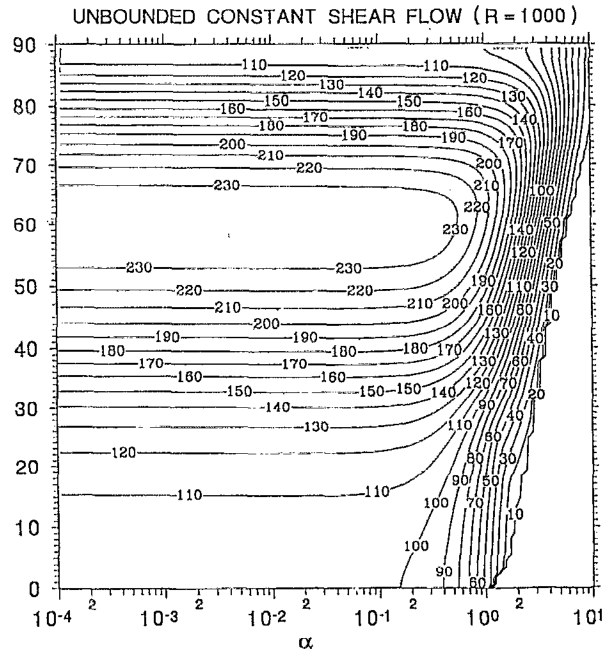

FIG. 1. Maximal energy growth, $G_{k,l}$, as a function of the streamwise (k), and spanwise (l) wave number over a time interval $T = 10$ and $R = 1000$ (based on $\alpha = 1$) for unbounded constant shear flow. The abscissa is $\alpha = (k^2 + l^2)^{1/2}$ and the ordinate is $\Theta = \tan^{-1}(l/k)$. The maximal growth occurs for $\Theta = 63°$. Note that for small wave-number viscosity does not affect the growth attained, and that for larger wave-number viscosity affects least the structures neighboring the streamwise rolls ($\Theta = 90°$).

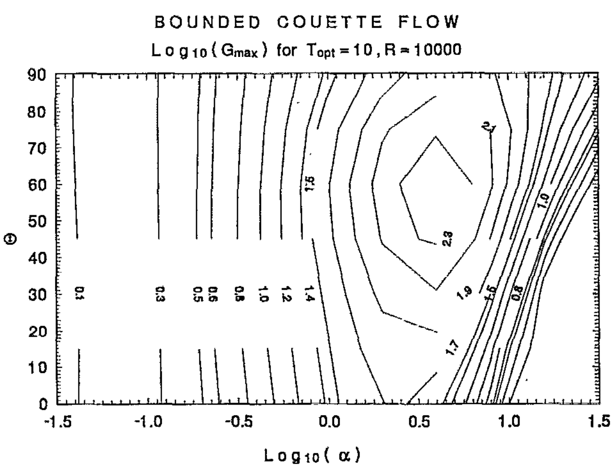

FIG. 2. Maximal energy growth, $\log(G_{k,l})$, as a function of the streamwise (k), and spanwise (l) wave number over a time interval $T = 10$ and for $R = 10\,000$ in Couette flow. The abscissa is $\alpha = (k^2 + l^2)^{1/2}$ and the ordinate is $\Theta = \tan^{-1}(l/k)$.

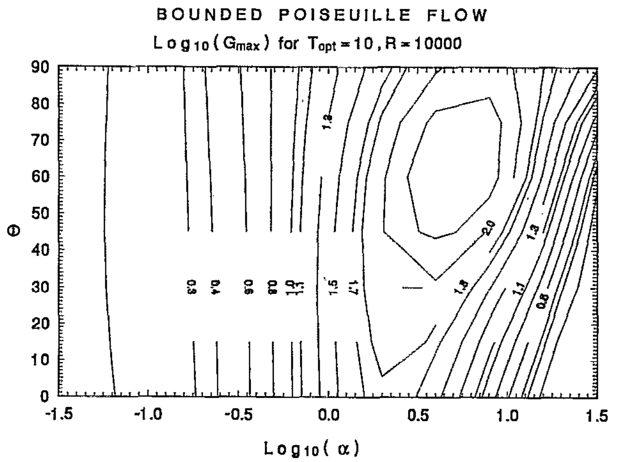

FIG. 3. Maximal energy growth, $\log(G_{k,l})$, as a function of the streamwise (k), and spanwise (l) wave number over a time interval $T=10$ for $R=10\,000$ in plane Poiseuille flow. The abscissa is $\alpha=(k^2+l^2)^{1/2}$ and the ordinate is $\Theta=\tan^{-1}(l/k)$.

While the normal modes of the dynamical operator for the canonical examples of viscous shear flow are stable and bear no clear relation to the structures observed either in experiments or in numerical simulations of turbulent shear flow, the optimally growing 3-D structures in these flows possess a universal character that is in accord with observations.[5,9] The structures that dominate growth over small multiples of the shear time scale are combinations of cross-stream waves and streamwise rolls and these are associated with a characteristic spectrum in the (k,l) plane. This spectrum comprises an axis at $k=0$ where the pure streamwise roll solution occurs, a 2-D shear wave axis at $l=0$ which corresponds to the cross-stream wave solutions of Orr,[12] and a maximum in the inertial subrange for oblique waves with $l/k \approx \tan(60°)$ (for an eddy turnover time of 10 advective units) for which a combination of the mechanisms of growth arising in these two limits produces optimal development. In this work the universal nature of the optimal solutions is revealed in the examples of the spectrum of Couette, plane Poiseuille, and unbounded shear flow.

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