

Optimal excitation of perturbations in viscous shear flow

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(Received 14 December 1987; accepted 29 April 1988)

Evidence, both theoretical and experimental, is accumulating to support a mechanism for transition to turbulence in shear flow based on the 3-D secondary instability of finite 2-D departures from plane parallelism. It is of central importance for using this mechanism to understand how the finite amplitude 2-D disturbances arise. To be sure, it is possible that in many experiments the disturbance is produced by the intervention of a mechanism that directly injects the requisite disturbance energy without calling on the store of kinetic energy inherent in the shear flow. It is shown here that it is also possible to tap the mean shear energy using properly configured perturbations that develop into the required primary disturbance on time scales comparable to those associated with the secondary instabilities even though the shear flow is stable or supports, at most, weak exponential instability.

I. INTRODUCTION

While the growth of small perturbations in shear flow is usually ascribed to linear modal instability, it is known that properly configured perturbations not of normal mode form undergo a period of transient growth resulting in large increases of amplitude and energy. For flows such as the Couette that do not support unstable normal modes this transient growth must account for whatever increase of perturbation energy is observed, assuming the initial disturbance is sufficiently small that linearity is maintained. For flows such as the Poiseuille that do support unstable normal modes the excitation of the unstable mode is determined by the configuration of the initial perturbation and exciting the mode in isolation is shown here to be highly suboptimal.

A consideration of energetics reveals that developing perturbations must have momentum fluxes directed down the mean momentum gradient and this can guide the choice of energetically active disturbances; with experience one can readily produce rapidly growing waves if the Reynolds number of the flow is sufficiently large. Still, it is worthwhile to find the optimal initial condition that most effectively excites a given mean flow. Such an optimal excitation allows an assessment of the potential of the initial value problem to account for the growth of perturbations in stable flow or to excite the unstable mode rapidly. Additionally, the structure of the optimum shows how a given mode is most easily excited; for instance, some modes are efficiently induced by boundary disturbances and others by interior disturbances.

In this work optimal initial conditions for the excitation of 2-D disturbances are found as solutions to a variational problem and the results illustrated with examples from the viscous Poiseuille and Couette flows: the optimum is found for the unstable Poiseuille mode and the least damped Couette mode. The constraint that a particular mode be optimized is then relaxed and the most rapidly growing disturbance in a given time is found without regard to projection.

The mechanism of transient growth is examined with the model problem of plane parallel viscous shear flow for which stationary solutions of the form $U(z)$ to the equations of motion in a channel bounded by horizontal planes at

$z = \pm 1$ are linear combinations of the two basic profiles of plane Poiseuille flow,

$$U(z) = 1 - z^2,$$

corresponding to a constant imposed pressure gradient and Couette flow,

$$U(z) = z,$$

corresponding to imposed plate motion $U(1) = 1.0$, $U(-1) = -1.0$. The linear evolution of perturbation streamfunction, $\psi = \psi(z)e^{ik(x-ct)}$ with $u = \partial\psi/\partial z$ and $w = -ik\psi$ is governed by the Orr-Sommerfeld equation

$$(ikR)^{-1} \left(\frac{\partial^2}{\partial z^2} - k^2 \right)^2 \psi = (U - c) \left(\frac{\partial^2}{\partial z^2} - k^2 \right) \psi - U_{zz} \psi, \quad (1a)$$

with boundary conditions expressing the vanishing of velocity at the rigid plates,

$$k\psi = \frac{\partial\psi}{\partial z} = 0, \quad z = \pm 1.0. \quad (1b)$$

It is understood in this linear problem that physical significance is ascribed to the real part of complex quantities.

This linear stability problem for viscous flow is characterized by the nondimensional Reynolds number $R = U_{\max} L / \gamma$, where γ is the kinematic viscosity, L is the half-channel width used to nondimensionalize distance, and U_{\max} is the maximum mean flow velocity providing the time scale L / U_{\max} .

The growth of small disturbances can arise either from modal instability or from the transient development of a properly configured initial perturbation. For Couette flow there are no unstable modes at any Reynolds number, a theoretical result¹ consistent with numerical experiments.² Transient growth in the linear Couette problem was examined by Orr³ who found that all perturbations decrease for $R < 44.3$, but above this value growth is possible. He was unable to determine the extent of the growth because of analytical difficulties associated with the viscous boundary conditions.

Poiseuille flow has no exponential temporal instabilities for $R < 5772.22$, according to Orszag.² Above this value

there is an unstable symmetric mode that obtains, for $R = 10^4$ and $k = 1.0$, the complex phase speed of $c = (0.238, 0.00374)$, corresponding to one e -folding in the time taken for 134 channel width advections by the maximum mean flow velocity. This growth rate is two orders of magnitude smaller than that typical of inflection point instabilities in inviscid shear flow, but the slow growth cannot be directly attributed to the effects of viscous dissipation at Reynolds numbers of 10^4 . As will be made clear by example, the slow growth rate of normal modes arises from modal structures that are highly inefficient at converting mean flow energy to perturbation energy.

The importance of the least stable normal mode derives from its being a long-lived structure in the linear problem. If one wishes to excite waves by drawing on the mean flow energy through Reynolds stress mediated conversion from mean energy to perturbation energy, then other things being equal it is best to call on a perturbation that excites the most persistent structure. Because the instability is so weak, it does not matter much over the first few hundred units of nondimensional time whether the structure is slowly growing or slowly decaying. Thus the qualitative distinction between the Poiseuille and Couette flow, that the former supports instabilities while the latter does not, will fail to produce a qualitative difference in the results. The Couette flow does possess a slightly damped mode at moderately large Reynolds number and behavior similar to that of the Poiseuille flow is found.

It is certainly true that the most unstable mode (if it exists) dominates the response of the flow in the limit $t \rightarrow \infty$ to any disturbance not deliberately configured to have a null projection on it. However, it is likely that many physical problems and experiments are appropriately analyzed as initial value problems with excitation confined in time and not controlled to project only on the single most unstable mode. If the additional modes of the problem that are unavoidably excited were irrelevant to the growth of perturbations then it would be permissible to ignore them; however, as will be demonstrated, this is not so. Even if the most unstable mode is dominant at some later time the initial setup of the mode will be shown to be inextricably linked to interactions with its companion stable modes.

The outline of the paper is as follows. In Sec. II the optimal excitation problem in the L_2 norm is solved. As this norm is perhaps not as physically motivated as the energy norm, the optimal excitation for the energy norm is also found. In Sec. III the requirement that a specific mode be optimally excited is relaxed and in its stead a time interval is imposed over which the perturbation is required to increase by the greatest possible amount. The motivation here is physical: we are attempting to find explosive growth scenarios. While these optima are not likely to be exactly duplicated by a stochastic forcing they do provide an upper bound on what growth can result from a perturbation. An interesting result emerges in that the potential for growth increases rapidly with Reynolds number above about 500. Stochastic forcing is not effective until this Reynolds number has been exceeded but above this value forcing can be highly effective in exciting disturbances.

II. OPTIMAL EXCITATION OF A MODE

The Orr–Sommerfeld equation together with its boundary conditions (1) can be written in operator notation, assuming the solution form $\psi(x, z, t) = E(z)e^{ik(x-ct)}$, as

$$\begin{aligned} LE(z) &= cE(z), \\ L &\equiv \Delta^{-1}(-\Delta^2/ikR + U\Delta - U_{zz}). \end{aligned} \quad (2)$$

Nontrivial solutions to (2) are associated with complex phase speeds c . In this work only flows confined between boundaries at finite z are used as examples. It has been shown that under this circumstance the spectrum of modes is discrete⁴ and complete.^{5,6} By contrast, Mack⁷ demonstrates by example that completeness in boundary layer flows that are unbounded above requires a continuous spectrum in addition to a set of discrete modes.

In the examples to follow, L is expressed in finite difference form.⁸ Various orders of finite differences and numbers of collocation points were tried and the resolution necessary to obtain converged solutions determined. Results shown used 100 grid points and seven point difference approximations. Eigenvalues extracted using the QR algorithm were compared with those of Orszag² and the first eigenfunction for Poiseuille flow at $R = 10^4$ with the tabulated values of Thomas.⁹

The N modal solutions in the finite difference approximation are each represented by a vector

$$\psi_j = E_j e^{ik(x - ct)}. \quad (3)$$

Assuming a fixed wavenumber k , we can express the evolution in time of an initial perturbation $\psi_0 e^{ikx}$ as

$$\psi = \sum_{j=1}^N \alpha_j E_j e^{ik(x - ct)},$$

where α is the spectrum of the perturbation obtained using E , which is the matrix having the eigenvectors as columns:

$$\alpha = E^{-1} \psi_0. \quad (4)$$

The concept of an optimum requires a measure for the perturbation magnitude. One choice is the rms amplitude of the streamfunction, which has the advantage of simplicity. A second and perhaps more physical measure is the rms amplitude of the streamfunction gradient corresponding to the rms velocity. The square of this norm is proportional to the kinetic energy. The former will be referred to below as the L_2 norm and the latter as the energy norm. An additional advantage gained by using two norms is that examples of optimal perturbations in these norms can be compared to obtain an impression of the general attributes of optima independent of the specific measure.

We now consider some example problems, the first posed as follows: find the minimum initial disturbance required to excite a chosen mode at unit amplitude. The motivation might be to find the best way to produce the most persistent wave, in that case the least stable mode would be the target. It might be supposed that the best way to excite the least damped wave would be to put the available amplitude or energy, as the case may be, directly into that mode. This is not so; in fact, exciting the desired mode directly is

highly suboptimal. It is much better in flows with nonorthogonal modes to distribute the initial disturbance so that the interaction between the nonorthogonal modes and the mean flow transfers energy from the mean to the perturbation. The result of such a choice of perturbation can be an increase in disturbance energy even for a model problem with only stable or damped modes. This disturbance energy is drawn from the mean flow despite the absence of instability.

With a little experience one becomes skilled at producing perturbations that are energetically active and the attributes of such disturbances will become clear from examples below. Limits on transient growth are determined by finding optimal perturbations. If no perturbation could produce significant growth for a Reynolds number below some value in a given flow, then the energy of the flow is not available in the linear limit. This is a quite different result from the absence of exponential instabilities placing a limit on $t \rightarrow \infty$ asymptotic growth, and the existence of such instabilities has no clear relationship with the potential for transient growth, as should be clear from examining the Couette and Poiseuille problems. In practice, transient development seems only to require that there be energy available in the mean flow as shear, or more generally as a deformation field,¹⁰ and that the Reynolds number be large enough. These are conditions required by general integral bounds¹¹ but whereas these theorems limit what might be done in theory, the optima provide concrete examples of what can be done in fact.

The solution to finding the best perturbation for exciting a given mode is found by a variational method. In the L_2 norm the functional to be minimized is $\psi^* \cdot \psi$. It is written using the spectrum (4) at $t = 0$ as

$$(\mathbf{E}\alpha)^* \cdot (\mathbf{E}\alpha) = \alpha^* \mathbf{E}^* \mathbf{E} \alpha = \alpha^* \mathbf{A} \alpha. \quad (5)$$

Here \mathbf{A} is the matrix of the positive definite quadratic form that associates with a spectrum α , the square of the L_2 norm of its streamfunction ψ .

Choosing as a constraint that the i th mode be of unit magnitude, the variational problem is to render stationary the function

$$F = \alpha^* \mathbf{A} \alpha + \lambda (\alpha^* \epsilon_i - 1),$$

where ϵ_i is the unit column vector. Setting the first variation in α to zero gives, recognizing \mathbf{A} to be Hermitian,

$$\mathbf{A} \alpha = -\lambda \epsilon_i.$$

The optimal spectrum

$$\alpha = -\lambda \mathbf{A}^{-1} \epsilon_i \quad (6)$$

is completed by choosing λ so that $\alpha_i = 1.0$.

There is a relationship between this optimum and the eigenvectors of the matrix adjoint to L . Whereas the eigenfunctions of a self-adjoint operator are orthogonal and consequently dynamically independent, a non-self-adjoint operator such as L has an associated adjoint operator L^* , its Hermitian transpose, with eigenvalues that are the complex conjugates of the eigenvalues of L . The eigenvector of an eigenvalue in the adjoint matrix is orthogonal to all eigenvectors of the original matrix except for the one with an eigenvalue conjugate to its own. It is clear from (5) and (6) that

the L_2 optimal initial condition for exciting a mode is the adjoint mode.

Taking advantage of the observation that the desired initial condition is the eigenvector of the adjoint matrix, the optimization problem can be solved for all modes simultaneously by eigenanalysis of the adjoint. Figure 1 shows the unstable mode of the Poiseuille problem at $R = 10^4$ and the optimal initial condition to excite this mode while Fig. 2 shows of the two degenerate least stable modes in the Couette flow at $R = 10^3$ that one trapped near the lower boundary and the optimum associated with it. The development of these optimal perturbations is shown in Figs. 3 and 4, respectively. In these figures and those to follow, the maximum value of the streamfunction as well as the normalized square of the L_2 norm are indicated at each time so that the resolution of the plot need not be compromised by a constant contour interval.

Optima in the energy norm for the examples above are obtained in a similar way using the perturbation energy K :

$$4\bar{K} = k^2 \psi^* \cdot \psi + \psi_z^* \cdot \psi_z.$$

This can be expressed using the spectrum α as

$$4\bar{K} = \alpha^* B \alpha, \quad (7)$$

$$B \equiv k^2 \mathbf{A} + \mathbf{E}_z^* \mathbf{E}_z,$$

where \mathbf{E}_z is the matrix with the eigenvector derivatives as columns.

The optimization proceeds as before with the solution

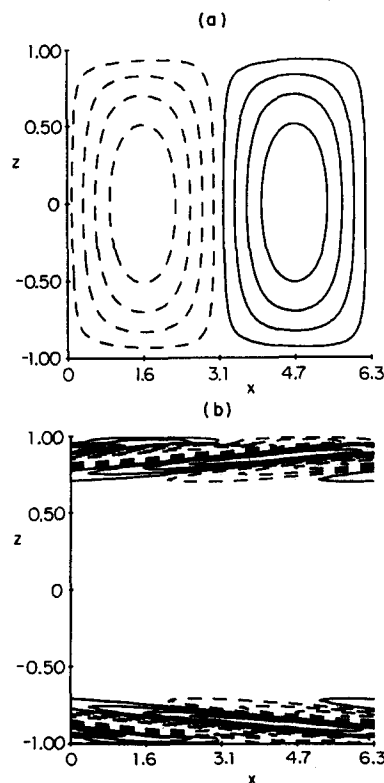


FIG. 1. Optimal excitation in the L_2 norm of the unstable eigenmode for Poiseuille flow with $k = 1.0$ and $R = 10^4$. (a) Perturbation streamfunction of the unstable eigenmode with eigenvalue $c = (0.378, 0.00374)$. (b) Disturbance streamfunction that is the optimal condition for exciting the unstable mode.

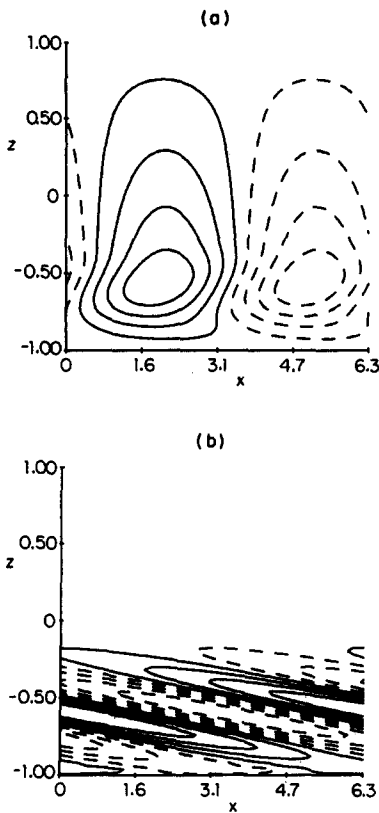


FIG. 2. Optimal excitation in the L_2 norm of the least stable mode for Couette flow with $k = 1.0$ and $R = 10^3$. (a) Perturbation streamfunction of the least stable eigenmode with eigenvalue $c = (0.605, -0.119)$. (b) Disturbance streamfunction that is the optimal initial condition for exciting the least stable mode.

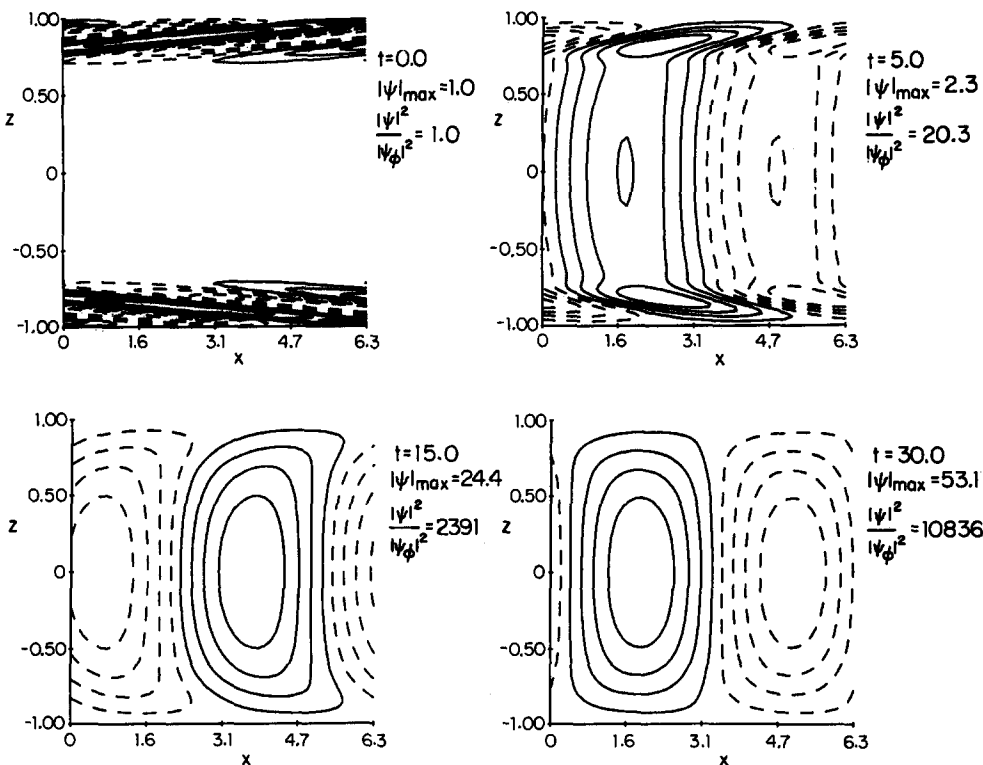


FIG. 3. Development of the perturbation streamfunction of the optimal L_2 norm initial condition for exciting the unstable eigenmode of Poiseuille flow with $k = 1.0$ and $R = 10^4$.

$$\alpha = -\lambda \mathbf{B}^{-1} \epsilon_i, \quad (8)$$

where λ is chosen to make the projection on the i th mode unity.

The minimum energy initial condition for the unstable mode in the Poiseuille problem at $R = 10^4$ is shown in Fig. 5 together with its development in time.

At this point we return to the adjoint matrix and examine a connection with the differential equation adjoint to the Orr-Sommerfeld equation:

$$(ikR)^{-1} \left(\frac{\partial^2}{\partial z^2} - k^2 \right)^2 \phi = \left(\frac{\partial^2}{\partial z^2} - k^2 \right) (U - c) \phi - U_{zz} \phi, \quad (9a)$$

with boundary conditions

$$k\phi = \frac{\partial \phi}{\partial z} = 0, \quad z = \pm 1.0. \quad (9b)$$

The argument leading to expression (6) for the optimal spectrum using the vector product $\psi^* \cdot \psi$ can be made in a similar fashion for continuous functions by replacing the vector product with the analogous expression

$$\int_{-1}^{+1} \psi \psi^* dz. \quad (10)$$

Properly normalized eigenfunctions of the Orr-Sommerfeld equation, ψ_i with eigenvalue c_i and of the adjoint equation ϕ_j with eigenvalue c_j , are biorthogonal in a way similar to that of their matrix analogs,^{5,12}

$$\int_{-1}^{+1} \psi_i \left(\frac{\partial^2}{\partial z^2} - k^2 \right) \phi_j dz = \delta_{ij}. \quad (11)$$

Integration by parts permits ψ and ϕ to be interchanged

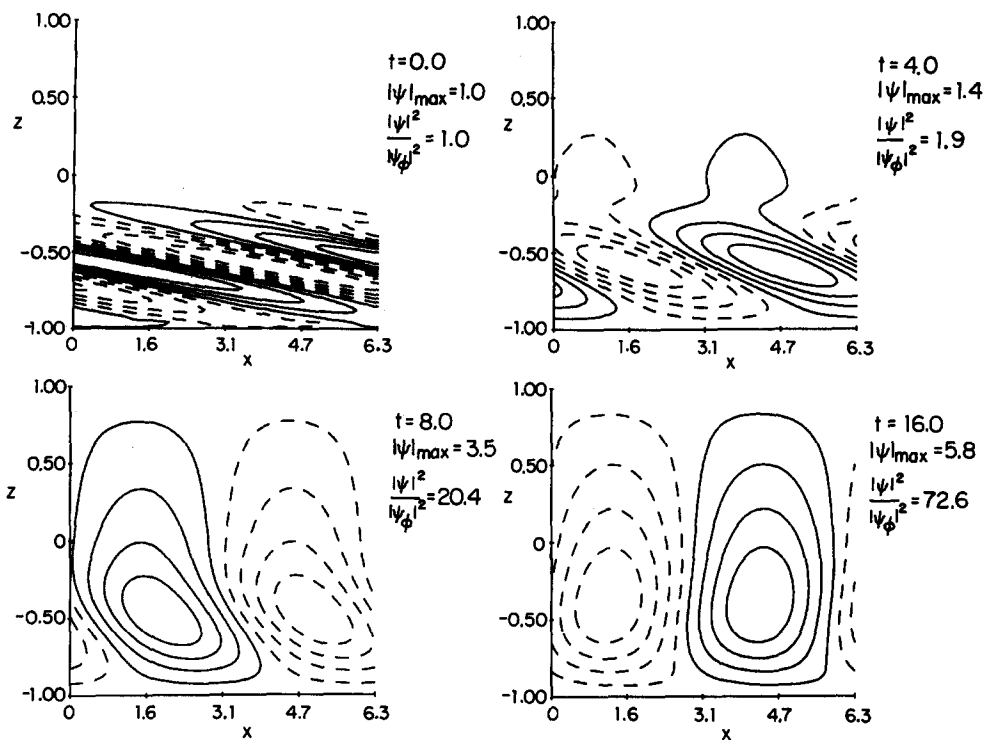


FIG. 4. Development of the perturbation streamfunction of the optimal L_2 norm initial condition for exciting the least stable eigenmode of Couette flow with $k = 1.0$ and $R = 10^3$.

in this expression. An examination of (8), (10), and (11) then shows that the analogy between the matrix optimal solution and the differential optimum requires the energy norm optimum be identified with the conjugate of the corresponding adjoint eigenfunction. In the L_2 norm the analogy would be complete if the matrix adjoint could be identified with the Laplacian of the conjugate of the differential adjoint. The optimal excitation for the mode ψ_i would then be $(\partial^2/\partial z^2 - k^2)\phi_i^*$. Unfortunately, while ϕ_i satisfies the boundary conditions appropriate for a streamfunction, its

Laplacian in general does not and it is required that the expression for the optimum (6) be solved using a generalized inverse. This is not a conceptual impediment, but, for our purpose, it suffices to observe that the Laplacian of ϕ_i^* is nearly identical to the matrix adjoint except for a region very near the boundary where the adjustment to the boundary conditions is accomplished.

The eigenanalysis of the adjoint equation (9) was performed for the Poiseuille flow example of Fig. 1 to illustrate the above argument. As expected, the eigenvalues are identi-

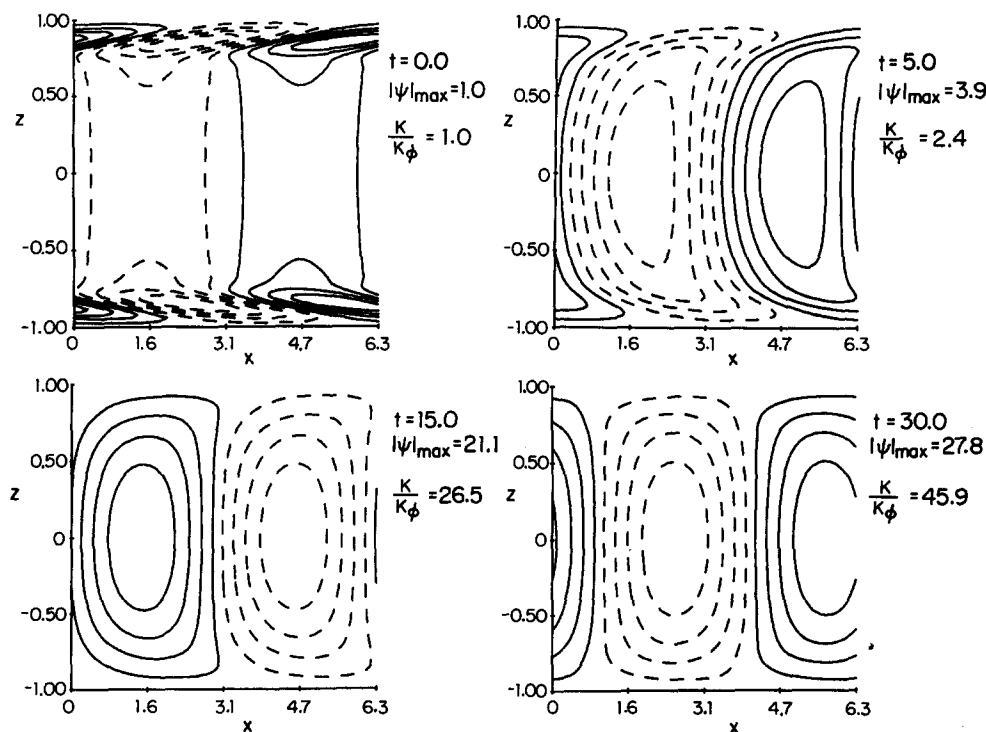


FIG. 5. Development of the perturbation streamfunction of the optimal energy norm initial condition for exciting the unstable eigenmode of Poiseuille flow with $k = 1.0$ and $R = 10^4$.

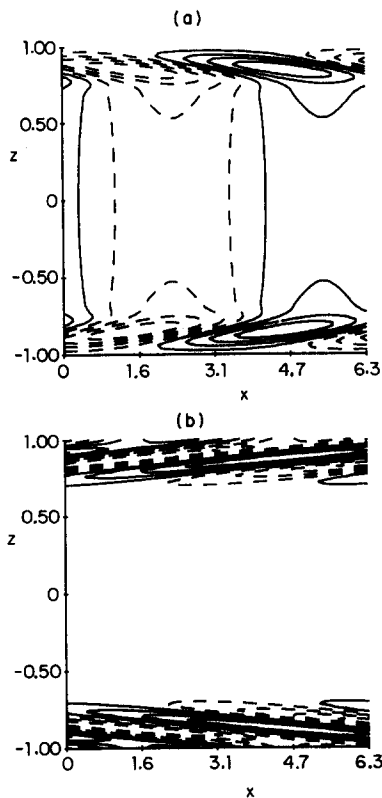


FIG. 6. Perturbation streamfunction of the eigenmode of the adjoint Orr-Sommerfeld equation, which is associated with the unstable eigenmode of Poiseuille flow illustrating the relation between this adjoint eigenmode and the optimal initial condition for exciting the unstable mode. (a) The adjoint Orr-Sommerfeld mode associated with the unstable eigenmode of Poiseuille flow for $k=1.0$ and $R=10^4$. The eigenvalue is $c=(0.378, 0.00374)$. (b) The Laplacian of the conjugate of the adjoint mode which closely approximates the optimal initial condition.

cal to those of (1), the energy norm optimum in Fig. 5(a) is identical to the conjugate of the adjoint in Fig. 6(a), and the approximate L_2 optimum deviates from the matrix solutions only in the immediate vicinity of the boundary. The unstable mode provides the severest test as its matrix adjoint solution is highly concentrated near the boundary precisely where the argument is likely to fail. Even so, the deviations are slight, as seen in Fig. 6 where the adjoint mode and its Laplacian are shown. Figure 6(b) is to be compared to Fig. 1(b) and the adjustment to the boundary conditions noted.

In summary, the energy norm optimum is the conjugate of the companion adjoint mode and a good approximation to the L_2 optimal excitation for a given mode is the Laplacian of the conjugate of its companion mode in the adjoint problem subject only to a smooth transition in the immediate vicinity of the boundary. Further development of the relation between the differential adjoint and the matrix optimal solutions is beyond our present scope and we return now to the question of identifying the most rapidly growing disturbances.

III. OPTIMAL GROWTH PERTURBATIONS

Another physically interesting problem results from relaxing the constraint that a given mode be optimally excited and instead seeking the perturbation that produces the maximum total growth over a time interval of a few tens of advection periods. The purpose is to identify initial conditions that grow explosively and to compare the development of these less constrained optima with that of the specific mode excitations. In addition, a revealing dependence of the growth on Reynolds number is found.

The functional to be rendered stationary is

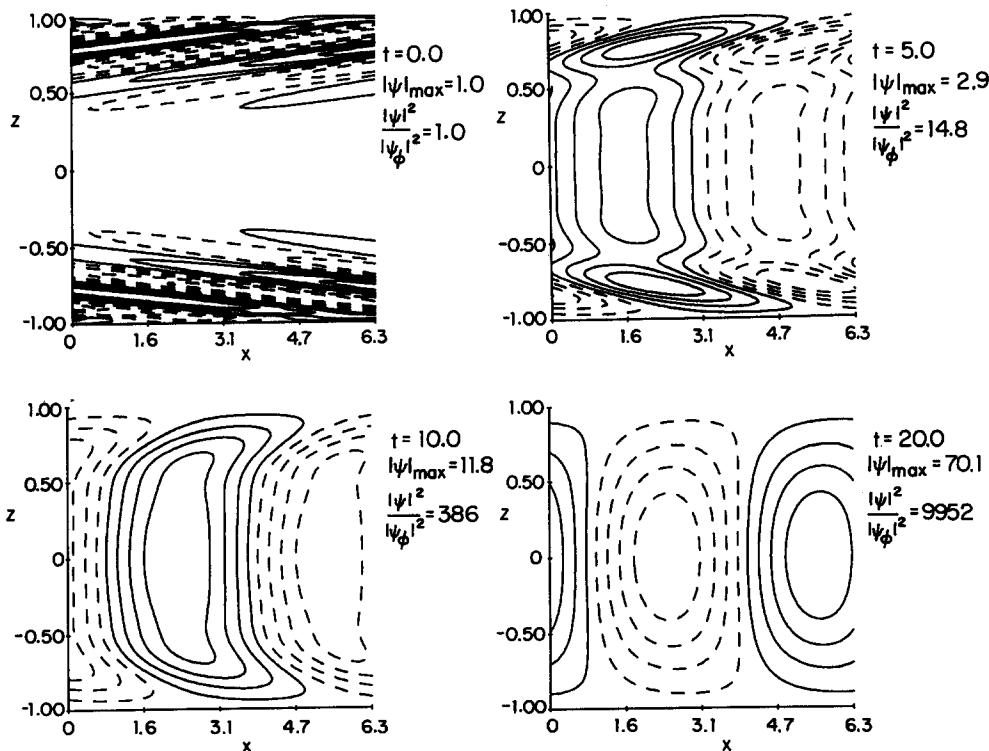


FIG. 7. Development of the perturbation that increases maximally in the L_2 norm over 20 time units for Poiseuille flow with $k=1.0$ and $R=10^4$.

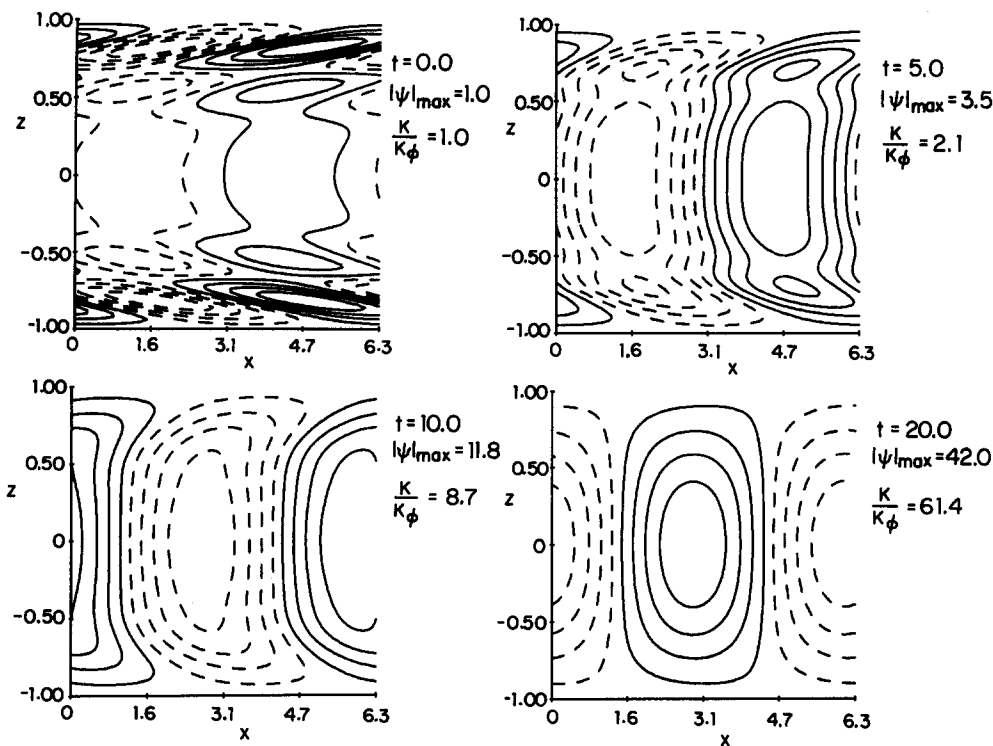


FIG. 8. Development of the perturbation that increases maximally in the energy norm over 20 time units for Poiseuille flow with $k = 1.0$ and $R = 10^4$.

$$F = \alpha^* A_t \alpha + \lambda (\alpha^* A_0 \alpha - 1).$$

Here $A_t \equiv E_t^* E_t$, where E_t is the matrix of eigenvectors each advanced in time according to (3).

The requirement of stationarity is

$$A_t \alpha + \lambda A_0 \alpha = 0,$$

$$(A_0^{-1} A_t + \lambda I) \alpha = 0.$$

The eigenvectors of the above matrix are the spectra of the stationary solutions, one of which is the desired optimum. As an example the optimum for an interval $t = 20.0$ and its development in time is shown in Fig. 7 for the Poiseuille with $k = 1.0$, $R = 10^4$. The same example optimized in the energy norm is shown in Fig. 8. Although the precise form of the optimal growth disturbance depends on the choice of norm, it is clear from Figs. 7 and 8 that there are general features common to favorable perturbations for exciting the unstable mode including upshear tilt in the phase lines and concentration near the boundaries. Other disturbances sharing these features can be expected to grow but to be suboptimal.

An energy optimum for the interval $t = 12.0$ in the Couette flow with $k = 1.0$, $R = 10^3$ is shown in Fig. 9. The total growth here is considerably less than that found in the previous two examples, revealing that perturbation growth is severely reduced for $R < 10^3$. Growth rates for perturbations optimized in the energy norm over the interval $t = 20$ for Poiseuille flow at $R = 500, 10^3, 10^4$ are shown in Fig. 10 to illustrate this point. Even so, these growth rates are comparable to the maximum rates associated with inviscid inflection point instabilities and are nearly two orders of magnitude greater than the growth rate of the viscous unstable mode at $R = 10^4$. If plotted in Fig. 10, the unstable mode

growth rate would not be distinguishable from the abscissa line.

IV. DISCUSSION AND CONCLUSIONS

The development of perturbations in viscous shear flow arises from the transfer of the kinetic energy of the mean flow to the perturbation scale mediated by a systematic down gradient flux of momentum by the perturbation Reynolds stress. A useful distinction can be made between two wave processes that produce this flux: exponential instabilities that dominate the long-time asymptotic limit in the linear problem because of their unbounded growth, and transient growth processes arising from energetic interaction between the modes and the mean flow because the modes are not mutually orthogonal. The first process cannot explain the growth of disturbances in stable flows such as the Couette. Even in flows that do support exponential instabilities, such as the Poiseuille, the second process is potentially able to dominate the development over time scales of a few hundred advection periods because of the slow growth of the unstable mode. A necessary first step in validating the hypothesis that transient growth processes be of importance is the demonstration here that properly configured disturbances can grow in viscous shear flows at a rate comparable to that of inviscid inflection point instabilities and can sustain this growth over sufficient time to realize total growth of nearly two orders of magnitude.

Because exponential instabilities in viscous parallel shear flow are either nonexistent or have small growth rate, it is now widely believed that rapid growth arises from secondary 3-D exponential instabilities.¹³ The transient development demonstrated here provides a mechanism for pro-

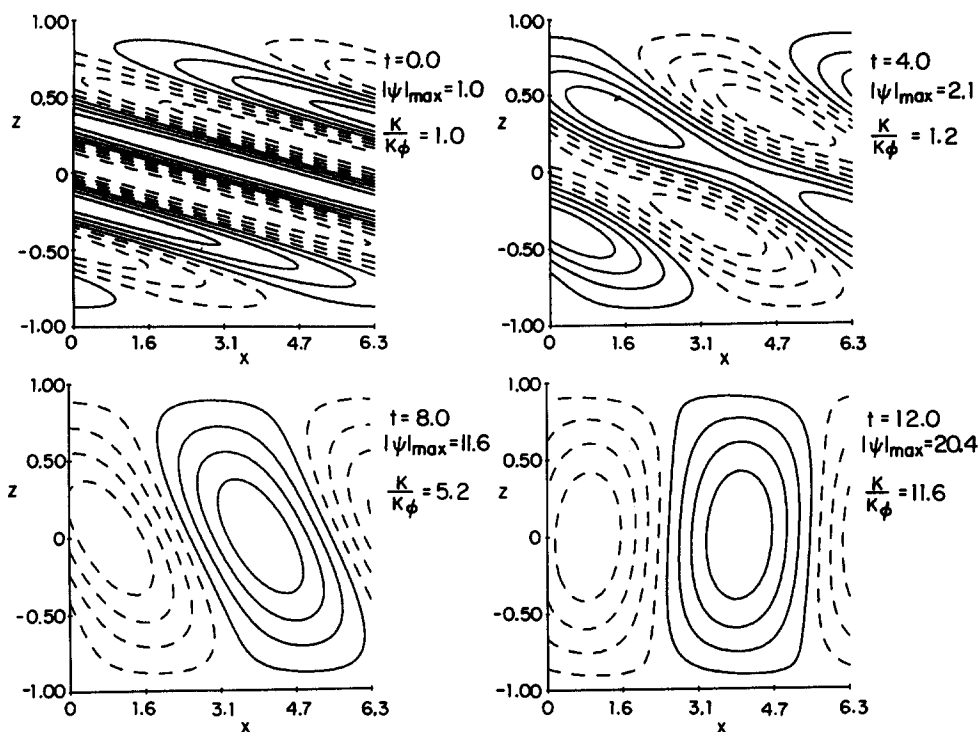


FIG. 9. Development of the perturbation that increases maximally in the energy norm over 12 time units for Couette flow with $k = 1.0$ and $R = 10^3$.

ducing from small initial disturbances the finite amplitude 2-D perturbed flow that supports 3-D cross-stream instabilities. However, it is also possible that the generation and maintenance of a highly perturbed flow is more essentially related to its perturbation spectrum and it is useful to distinguish a weak and strong version of this hypothesis. The weak version recognizes that the conceptual basis of exponential mode dominance is valid only in the limit of long time because it requires the most rapidly growing mode to obtain dominance over both the energetically active neutral and damped waves as well as over competing instabilities. For every perturbation that effectively excites an exponential mode, as in the above examples, there is an equivalently inefficient perturbation that, rather than advancing the setup of the instability inhibits it. Because of the exponential character of the growth, the perturbation with favorable origins

will be increasingly dominant. The importance of the initial condition in this weak version derives from its role in determining the connection between the statistical distribution of perturbations and the resulting field of saturated instabilities as they equilibrate nonlinearly. The strong version arises on noting that exponential dominance is not assured over time scales compatible with at least some experiments; thus it is possible that the instability achieves normal mode form after the greater part of its development has occurred and that the individual energetic event is primarily of the transient type discussed above. In either case the normal mode structure is found after the development is completed but comparison of time scales for exponential growth with time scales characteristic of an energetic event should allow a discrimination between these processes. There does not appear to be a justification for assuming that perturbations are of normal mode form to begin with and once it is seen that the configuration of the perturbation is crucial in determining its development, it is clear whether the weak or strong version is operating in a particular case that the transient development cannot be ignored. In some experiments the form of the initial disturbance is at least partially controlled and the observation here that disturbances introduced near the boundaries are particularly favorable in producing rapid growth is compatible with experimental results involving trip wires, excitation ribbons, spark gaps, and the like that are often deployed in those regions. It is also interesting that the least damped mode, the persistent structure in the linear problem, arises naturally both as the result of optimal excitation and in experiments.

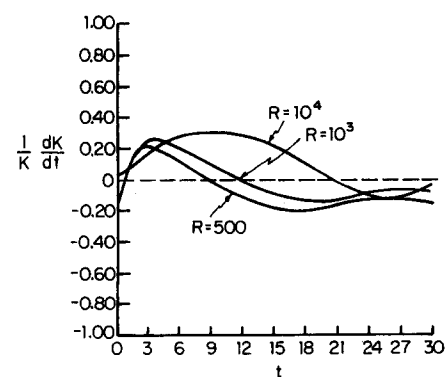


FIG. 10. Energy growth rate at the indicated Reynolds numbers in Poiseuille flow for perturbations optimized on an interval of 20 time units with $k = 1.0$.

Although the results presented here address the viscous problem governed by the Orr-Sommerfeld equation, the interior flow far from boundaries is in many respects similar to the inviscid problem. Practically speaking, this similarity

underlies the historical success of inviscid dynamics in applications. The relation between the transient growth phenomena and the stable and unstable modes has been recently developed for inviscid shear flows and qualitatively similar phenomena to those described here are found, including rapid initial growth at rates exceeding exponential mode rates and the preferential setup of a linear normal mode as a persistent structure.¹⁴⁻¹⁶ At higher amplitude the equilibrated structure are nonlinear modes.¹⁷ A preferred inviscid example in the work just referenced is the Eady problem of baroclinic instability theory. This problem can be identified with the Couette problem with free boundaries and has the important theoretical advantage of supporting normal modes, the linear persistent structures, and of being solvable exactly.

For plane Couette flow the growth of perturbations can only arise in the linear approximation from the mechanism addressed here as the problem does not support unstable modes. It might appear that for $R > 5772.22$ the Poiseuille flow, which does support an instability, would exhibit finite amplitude perturbations starting from an arbitrarily small initial disturbance if one is willing to wait long enough. In fact this is only true for experimental apparatus with periodic boundary conditions in the along-stream direction without which Poiseuille flow has been maintained laminar by careful control of perturbations up to $R = 9000$, far above the instability boundary.¹⁸ This result arises from the fact that it is not sufficient for a temporal exponential instability to exist in an apparatus of finite along-stream extent for that flow to support unbounded growth in the linear limit, it is necessary additionally for the instability to have a vanishing group velocity so that the growing wave packet does not propagate out of the experimental domain leaving behind an undisturbed flow. If it is assumed that the apparatus is fixed in the laboratory, the only relevant instability in the limit of long time is the one with zero group velocity. This concept is well known in the theory of plasma instabilities¹⁹ where the existence of zero group velocity unstable modes is associated with the so-called absolute instability as opposed to the case when all unstable modes propagate away, referred to as convective instability. The group velocity of the unstable modes in Poiseuille flow has been found by Deissler²⁰ to be strongly positive. This result forbids unbounded growth in a finite apparatus and requires that the disturbance be continuously reexcited at the upstream end, which accords with the common use in experiments of trip wires, ribbons, and the like to excite the modes. Convective instability of Poiseuille flow makes the excitation central in producing the disturbance,²¹ while this work has demonstrated that there are perturbations especially effective for this purpose.

The mechanism of transient growth can account for the development of perturbations in flows other than plane parallel shear where the importance of the transient dynamics is clearly motivated by the small growth rates of the instabilities. Once a finite amplitude 2-D wave arises, the 3-D instabilities that it supports are often found to grow at rates within small factors of the maximum permitted by integral bounds, thus suggesting an early dominance of the unstable mode. It remains true, however, that 2-D transient growth is equally rapid over its shorter time scale and furthermore

there is no reason to assume normal mode form for the 3-D perturbations if nonmodal configurations have comparable growth rates. The dominance of exponential modes is plausible for time scales sufficiently long to justify ignoring the transient development, and this depends on the experiment, but even if the life cycle energetics are dominated by the exponential growth phase, the weak hypothesis described above points out a role for the transients. It is a matter only of increased computational complexity to calculate the modes of a higher-dimensional flow and considering that the dynamical equations are not self-adjoint it follows that the modes are not orthogonal and as a result have transient dynamics analogous to that found here.

Consider a stationary solution to the 2-D Navier-Stokes equations such as plane parallel shear flow or a nonlinear equilibrated wave such as the 2-D upper branch solutions of Poiseuille flow.¹³ In the comoving coordinate system stationarity requires

$$J(\bar{\psi}, \Delta \bar{\psi}) = (1/R) \Delta^2 \bar{\psi},$$

where $\bar{\psi}$ is the streamfunction of the stationary solution. Perturbations to this solution evolve on a rapid advective time scale associated with the Jacobian followed by a slow diffusive relaxation. If there is superimposed a field of small perturbations ψ' , the analysis here demonstrates for the plane parallel flow and suggests by analogy for 2-D flow that certain of these perturbations are dangerous in that they result in the explosive growth of disturbance energy. The picture that emerges is of a flow close to a stationary solution but with intermittent energetic events associated with the chance occurrence of dangerous perturbations. As the perturbations disrupt the solution on the advective time scale it is likely that diffusive equilibrium is never exactly obtained.

This preliminary study has left many issues unresolved, including the behavior of a disturbance localized in the along-stream direction. Using Fourier synthesis this can be thought of as a packet of waves. Strictly speaking, the above argument concerning the potential importance of the growth of the individual modes must be confirmed for a local packet made up of such waves if this corresponds to the experiment to which the theory is applied. If the modes are strongly dispersive the maximum local amplitude obtained could be reduced considerably. There are at least two pieces of indirect evidence relevant to this question: in examples of inviscid shear flow where analytic solutions can be obtained the dispersion of the packet is small over the time of the transient growth and the packet grows at the rate of the central mode¹⁵; also in examples of boundary layer flows Gaster and Grant²² observed the packet to maintain its integrity even in the nonlinear limit.

ACKNOWLEDGMENT

This work was supported in part by National Science Foundation Grant No. ATM-8712995.

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