

## Excitation of Nearly Steady Finite-Amplitude Barotropic Waves

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### ABSTRACT

We obtain an exact nonlinear stationary solution for barotropic waves in a  $\beta$ -plane channel and show that it can be excited under a range of initial conditions. Results show that a finite-amplitude wave in a constant shear flow, given an initial phase tilt against the shear and a sufficient initial amplitude, interacts with the mean flow to produce a nearly steady state close to the exact stationary solution. This equilibration process involves nonlinear transients; in particular, as the flow equilibrates, the emergence of critical levels is accompanied by the neutralization of local mean vorticity gradients at these levels, thus allowing the solution to attain a nonsingular modal structure.

### 1. Introduction

Persistent, organized, large-amplitude flow patterns are often observed in the atmosphere and the ocean; atmospheric blocks (Rex, 1951), cutoff highs and lows (Palmén and Nagler, 1949), and Gulf Stream rings and meanders (Richardson, 1983; Watts, 1983) are examples. Theoretical investigations of these quasi-steady features have generally fallen into several categories: (i) the construction of shape-preserving wavelike or isolated solutions (which are often unstable) without considering how such states are formed (e.g., Craig, 1945; Kuo, 1959; Larichev and Reznik, 1976; Flierl et al., 1980; McWilliams, 1980), (ii) the time-evolution of small disturbances into quasi-steady, large-scale coherent features which persist for long periods, implying possible stationary states which are not identified (e.g., McWilliams, 1984), (iii) the forcing of large-scale features (e.g., Shutts, 1983; Pierrehumbert and Malguzzi and Malanotte-Rizzoli, 1984, 1985; Read, 1985), and (iv) the formation of steady features by KDV wave dynamics (Malguzzi and Malanotte-Rizzoli, 1984, 1985). The purpose of this work is to provide a simple example illustrating the connection between initial-value solution and the stationary solution for a barotropic channel flow and to show that quasi-steady states can be achieved from a range of initial conditions in the absence of direct forcing.

The equilibration process considered in this work is closely related to the problem of wave-mean flow interaction, which has been the focus of many recent studies, especially for flows with critical levels. However, despite this attention, the development of a Rossby wave critical level is still a matter of controversy. Bretherton (1966) argued that the presence of a critical level in the fluid excludes the existence of stable modes unless the potential vorticity gradient vanishes at the critical level, a circumstance which he dismissed as an "accident of analysis." Interestingly, the quasi-

linear calculation of Geisler and Dickinson (1974) for a forced Rossby wave showed that the effect of the wave-mean flow interaction is precisely to erase the vorticity gradient at the critical level. Benney and Bergeron (1969), Warn and Warn (1976), and Béland (1976) have further examined the role of nonlinearity near a critical level; McIntyre (1982) and Killworth and McIntyre (1985) have pointed to the role of the transients on the mean flow; and recently Schoeberl and Lindzen (1984) have shown that barotropic instability in the absence of dissipation can induce the vorticity gradient to vanish at a critical level. The implication of these studies is that the flow could equilibrate, providing that some process is present to ensure a nonsingular modal behavior at critical levels; the precise nature of this process is, however, not well understood.

In this study, we offer a simple example of nonlinear equilibration of a barotropic flow in a  $\beta$ -plane channel. We first obtain an exact, analytic, stationary solution to this model problem. We then demonstrate numerically that, under a range of initial conditions, the initial-value solution does in fact approach the stationary solution, and as the flow equilibrates, the mean vorticity gradient is destroyed by the transients at all critical levels.

The analytic solution is presented in Section 2, numerical results are discussed in Section 3; and conclusions are summarized in Section 4.

### 2. An exact shape-preserving nonlinear Rossby wave solution

The nondimensional vorticity equation governing barotropic flows in a  $\beta$ -plane channel is

$$\frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, \nabla^2 \psi + \beta y) = 0, \quad (1)$$

where  $t$  is time;  $\nabla^2$  is the Laplacian in Cartesian coordinates  $(x, y)$ ,  $x$  and  $y$  are the along-channel and

cross-channel axes, 'respectively;  $\psi(x, y, t)$  is the streamfunction which defines the nondivergent velocity field  $(u, v) = (-\psi_y, \psi_x)$ ;  $J$  is the Jacobian; and  $\beta$  is the gradient of the Coriolis parameter in the  $y$ -direction. The nondimensional variables are scaled as follows:  $(x, y)$  by  $L$ , the channel width;  $(u, v)$  by a characteristic flow speed  $U$ ;  $t$  by  $LU^{-1}$ ; and  $\beta$  by  $UL^{-2}$ . The boundary conditions are  $\psi(0, y, t) = \psi(1, y, t)$  (periodicity in  $x$ ) and  $\psi_x(x, 0, t) = \psi_x(x, 1, t) = 0$  (free-slip rigid side walls); the periodicity condition is imposed so as to allow comparison with numerical results in Section 3.

For shape-preserving solutions, we assume  $\psi(x, y, t) = \psi(x - ct, y)$ , where  $c$  is the zonal phase speed of the wave. Let  $\Psi$  be the streamfunction in a reference frame moving with the wave,  $\Psi(x, y) = \psi(x - ct, y) + cy$ , then (1) becomes

$$J(\Psi, \nabla^2\Psi + \beta y) = 0; \tag{2}$$

the solution is therefore stationary in the wave coordinates.

The total streamfunction may be written as  $\Psi(x, y) = \bar{\Psi}(y) + \Psi'(x, y)$ , where an overbar denotes the zonal average and a prime the departure therefrom. A perturbation of the form  $\Psi'(x, y) = A'f(y) \sin 2\pi kx$  (where  $A'$  is the wave amplitude and  $k$  the zonal wave number) then satisfies the periodic boundary condition in  $x$  for integral values of  $k$ . Upon substitution, (2) leads to two conditions for determining the wave structure  $f(y)$  and the mean flow profile  $\bar{u}(y) = -\bar{\psi}_y$ ; i.e.,

$$ff_{yyy} - f_y f_{yy} = 0, \tag{3}$$

$$f_{yy} + (2\pi l)^2 f = 0, \tag{4a}$$

where  $l$  is the latitudinal wavenumber

$$(2\pi l)^2 = \left[ \frac{\beta - \bar{u}_{yy}}{\bar{u} - c} - (2\pi k)^2 \right], \tag{4b}$$

which can, at most, be a function of  $y$ . Multiplying (4a) by  $f_y$  and its derivative by  $f$  reduces (3) to

$$f^2(l^2)_y = 0.$$

The condition for a nontrivial solution is, therefore,  $l = \text{constant}$ . A solution which satisfies (3), (4), and the boundary condition  $\Psi_x(x, 0) = \Psi_x(x, 1) = 0$  is

$$f(y) = \sin 2\pi ly, \quad l = \frac{n}{2}, \tag{5}$$

where  $n = 1, 2, 3, \dots$ .

From the dispersion relation (4b) we obtain

$$\bar{u}_{yy}(y) + (2\pi\alpha)^2 \bar{u}(y) = \beta + \alpha(2\pi\alpha)^2, \tag{6}$$

where  $\alpha$  is the total wave number,

$$\alpha = (k^2 + l^2)^{1/2}. \tag{7}$$

Upon integration, (6) gives  $\bar{u}(y)$  in terms of its boundary values  $\bar{u}(0)$  and  $\bar{u}(1)$ . Integrating  $\bar{u}(y)$  then determines  $\bar{\Psi}(y)$  to within a constant.

The solution which emerges from this procedure has its vorticity proportional to the streamfunction, i.e.,  $\nabla^2\Psi = -(2\pi\alpha)^2(\Psi + \text{constant})$ , so that the Jacobian vanishes in the wave coordinates. This type of barotropic, steady wave solution has been discussed by Craig (1945) and Neamtan (1946). Essentially, a free wave, regardless of its amplitude, does not interact with the mean flow if it has the same total wave number as the mean flow, and by extension any linear combination of such waves is also a solution. We shall, however, use a single wave solution as an illustration throughout our discussion.

After some algebraic reductions, the solution for a single wave may be written as

$$\psi(x, y, t) = \bar{\psi}(y) + \psi'(x, y, t), \tag{8a}$$

$$\begin{aligned} \bar{\psi}(y) = & \frac{\bar{u}(1) - \bar{u}(0)}{4\pi\alpha \sin\pi\alpha} \left[ \cos 2\pi\alpha \left( y - \frac{1}{2} \right) - \cos\pi\alpha \right] \\ & - \frac{\bar{u}(1) + \bar{u}(0) - 2K}{4\pi\alpha \cos\pi\alpha} \left[ \sin 2\pi\alpha \left( y - \frac{1}{2} \right) + \sin\pi\alpha \right] \\ & - Ky + \bar{\Psi}(0), \end{aligned} \tag{8b}$$

where,

$$\begin{aligned} K = c + \frac{\beta}{(2\pi\alpha)^2} = & \left[ \frac{\tan(\pi\alpha) \bar{u}(1) + \bar{u}(0)}{2\pi\alpha} \frac{2}{2} \right. \\ & \left. + \bar{\psi}(1) - \bar{\psi}(0) \right] \left[ \frac{\tan(\pi\alpha)}{2\pi\alpha} - 1 \right]^{-1}; \end{aligned} \tag{8c}$$

$$\begin{aligned} \psi'(x, y, t) = & A' \sin 2\pi k(x - ct) \sin 2\pi ly, \\ \left( l = \frac{n}{2}, n = 1, 2, 3, \dots \right). \end{aligned} \tag{8d}$$

Apart from the arbitrary reference value  $\bar{\Psi}(0)$ , which shall be taken to be zero, (8) is determined by four constants,  $\bar{u}(0)$ ,  $\bar{u}(1)$ ,  $c$ , and  $A'$ , which are constrained by a given initial state. For an evolving flow governed by (1), with  $\psi_x$  identically zero at horizontal boundaries, the zonally averaged vorticity flux and the mean-flow acceleration vanish at the boundaries. In consequence  $\bar{u}(0)$  and  $\bar{u}(1)$  do not depart from their initial values. The conservation of momentum (i.e.,  $\bar{\psi}(1) - \bar{\psi}(0) = \text{constant}$ ) provides a constraint on the value of  $c$ , as given in (8c); thus the mass flux rather than the phase speed may be regarded as an external parameter. Finally, the total energy of the initial state provides an upper bound on the value for  $A'$ .

The exact nonlinear steady wave solution (8) is, in fact, a generalization of the classical Rossby wave solution in a uniform current, which can be recovered by setting  $\bar{u}(y) = \text{constant}$  in (8). The effect of  $\beta$  is simply to modify the phase speed in (8c), in accord with the familiar Rossby wave dispersion relation. Notice also that since the mean flow  $\bar{u}(y)$  is independent of the wave amplitude, its energy corresponds to the *minimum* energy in the initial state necessary for achieving this stationary solution, i.e.,

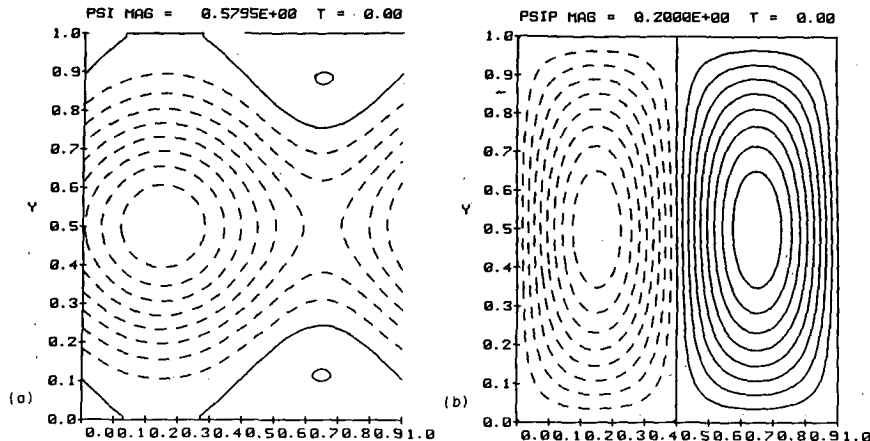


FIG. 1. The streamfunction for the gravest stationary mode according to (8). The parametric values used are  $\bar{u}(0) = -1/2$ ,  $\bar{u}(1) = 1/2$ ,  $\bar{\psi}(1) - \bar{\psi}(0) = 0$ ,  $A'_0 = 0.2$ , and  $\beta = 0$ . (a) The total streamfunction  $\psi$ ; (b) the perturbation streamfunction  $\psi'$ . The solid contours indicate positive values, and broken contours, negative values. The field is normalized with respect to the extremum magnitude shown at the top of the figure, and the contour spacing corresponds to 10 equal-interval contours between zero and the extremum.

$$E_{min} = \bar{E}(\alpha) = \int_0^1 \bar{u}^2(y) dy = \left[ \frac{\bar{u}(1) - \bar{u}(0)}{2} \right]^2 + \left[ \frac{\bar{u}(1) - \bar{u}(0)}{2 \sin 2\pi\alpha} \right]^2 \left[ \frac{1}{2} - \frac{\sin 2\pi\alpha}{4\pi\alpha} \right]. \quad (9)$$

For illustration, we consider an example where the total mass flux of the system corresponds to a Couette profile initially, i.e.,

$$\bar{\psi}(1) - \bar{\psi}(0) = - \int_0^1 \bar{u}_y|_{t=0} dy = \frac{-\bar{u}(1) + \bar{u}(0)}{2},$$

then (8) gives

$$\bar{u}(y) = \frac{\bar{u}(1) - \bar{u}(0)}{2 \sin 2\pi\alpha} \sin 2\pi\alpha \left( y - \frac{1}{2} \right) + \left[ c + \frac{\beta}{(2\pi\alpha)^2} \right], \quad (10a)$$

where

$$c + \frac{\beta}{(2\pi\alpha)^2} = \frac{\bar{u}(1) + \bar{u}(0)}{2}. \quad (10b)$$

Figures 1 and 2 show the structure of the gravest mode ( $k = 1, l = 1/2$ ) for  $\beta = 0$ . For display purposes, we have chosen  $\bar{u}(0) = -1/2$  and  $\bar{u}(1) = 1/2$ , which is consistent with a symmetric initial Couette profile. Figure 1a plots the total streamfunction for  $A' = 0.2$  (an arbitrary choice), and Fig. 1b gives the perturbation streamfunction. Since (1) is Galilean invariant, this symmetric solution is not physically different from one in which the initial flow is a westerly jet [ $\bar{u}(0) = 0, \bar{u}(1) = 1$ ], in which case (8) gives the more familiar pattern of an intense low lying southwest to a weak high, an analogue for blocking in a barotropic model.

Figure 2 shows that the solution for  $\bar{u}(y)$  has three critical levels for  $\psi'$ . Note in (10) that while  $\bar{u}(y)$  is

independent of  $\beta$ ,  $(\bar{u} - c)$  is not; a nonzero value of  $\beta$  would therefore break the symmetry of the critical levels shown in Fig. 2. For atmospheric applications, a typical dimensional value of beta is  $2 \times 10^{-11} \text{ (m s}^{-1})^{-1}$  and  $U \approx 30 \text{ m s}^{-1}$ , then  $\beta \approx 0.7$  for a 1000 km wave, or  $\beta \approx 6$  for a 3000 km wave; the corresponding critical-level displacements in Fig. 2 would be  $\Delta y \leq 0.01$ , hence negligible.

Interestingly, the mean vorticity gradient,  $\beta - \bar{u}_{yy}$ , is identically zero at all critical levels, as demonstrable from (6); hence the Bretherton (1966) condition for

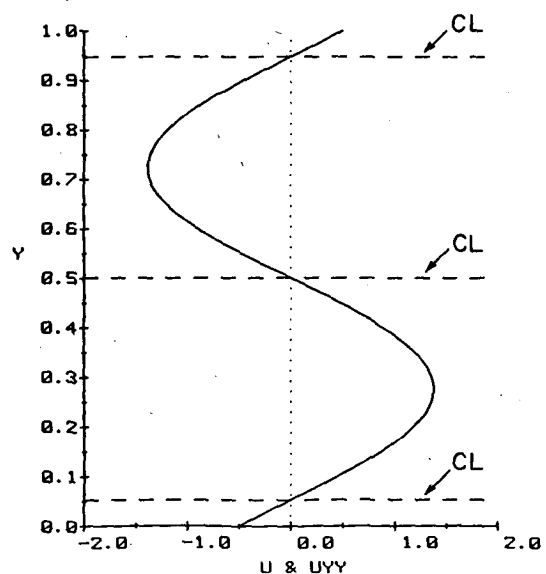


FIG. 2. Analytic solutions for  $\bar{u}(y)$  and  $\bar{u}_{yy}$ , scaled by  $-(2\pi\alpha)^2$  for the gravest stationary mode shown in Fig. 1. The broken lines labeled "CL" indicate the positions of the critical levels.

critical-layer stability is automatically satisfied. However, the flow *could* still be unstable to small perturbations—given that  $\bar{u}$  satisfies neither Rayleigh-Fjørtoft sufficient conditions for stability nor the more stringent stability condition of Arnol'd (1965) since  $\bar{u}/\bar{u}_{yy} < 0$  and its absolute magnitude,  $(2\pi\alpha)^{-2}$ , is less than  $(2\pi)^{-2}$  [see Eq. (13) of Arnol'd]. Furthermore, the steady-wave pattern may itself be unstable since the boundary conditions are zonally symmetric (see Andrews, 1984). It therefore remains to show that, despite its susceptibility to instabilities, this steady wave could persist for extended periods.

Moreover, the analytic solution (8) permits a multiplicity of steady waves. There is, of course, no obvious reason for preferring the gravest mode over the others. Thus, by (8) we have shown that there exists an exact, shape-preserving, finite-amplitude steady solution, but not that it is, in fact, the asymptotic limit to some initial perturbation. To show this, we appeal to numerical experiments.

### 3. Equilibration of finite-amplitude barotropic waves in shear

For an inviscid Couette flow, it is well known that a linear wave with a phase tilt against the shear of the mean flow can grow at the expense of the mean flow and the initial tendency is to turn  $\bar{u}(y)$  into an "S" profile (Orr, 1907; Farrell, 1982); i.e., it decelerates the mean flow in the upper half of the wave and accelerates below. In the light of Section 2, this suggests that such an initial wave configuration could give rise to the gravest stationary mode shown in Figs. 1 and 2. To investigate the evolution of the initial-value solution, we have constructed a grid-point barotropic vorticity model based on the Jacobian (Arawaka, 1966); the streamfunction is determined diagnostically using a fast Fourier transform in  $x$  and solving the structure in  $y$ .

Unless specified otherwise, the results which follow are based on  $64 \times 64$  square grids.

The central questions which concern us are: Does the initial-value solution approach the exact analytic stationary solution? If so, under what initial conditions? For such experiments, we consider an initial condition of the form

$$\bar{\psi}(y, 0) = -\frac{y}{2}(y - 1),$$

$$\psi'(x, y, 0) = A_0 \sin 2\pi(k_0 x + l_0 y) \sin(\pi y), \quad (11)$$

where the subscript "0" denotes an initial value at  $t = 0$ . In the following experiments, we take  $k_0 = 1$ ,  $l_0 = 1$ , and a series of initial wave amplitudes:  $A_0 = 0.01, 0.075, 0.10, 0.15, 0.20$  and  $0.25$ , which, in terms of energy, correspond, respectively, to  $E'_0/E_0 = 3\%, 60\%, 73\%, 86\%, 91\%$  and  $94\%$ , where  $E$  is the global average of  $(\psi_x^2 + \psi_y^2)$  and  $E'$  is similarly defined for the perturbation.

#### a. Small amplitude results

In cases where  $A_0 \leq 0.1$ , the solution does not evolve into a state with steady finite-amplitude waves. A common trait of this class of solution is that the wave ultimately decays, leaving a  $\bar{u}$  which is the mirror image of "S." Although this "reverse S" profile retains its coherence for many time units, the wave continues to be sheared as in the linear Couette problem.

To illustrate the evolution of  $\bar{u}(y, t)$ , Fig. 3 displays the time sequence for  $A_0 = 0.075$ . Initially, with a phase tilt opposite to the mean-flow shear, the wave extracts energy from the mean flow, and the mean-flow acceleration due to the growing wave is negative in the upper half of the wave and positive below, leading to the "S" profiles in Fig. 3a. The initial tendency of the mean flow is, therefore, toward the gravest stationary mode,

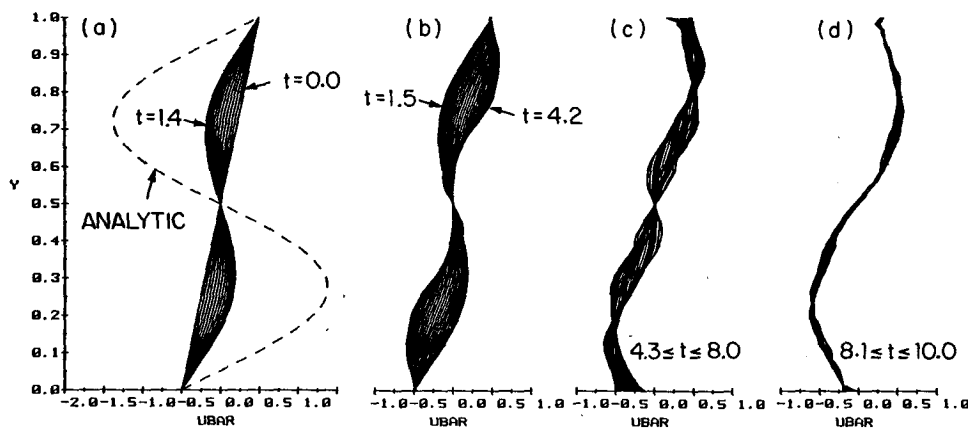


FIG. 3. The time evolution of  $\bar{u}$  for the case of  $A_0 = 0.075$ . The interval of the plots is 0.1 nondimensional time units, and the broken line is the analytic solution for the gravest stationary mode given by (10). The results for  $t > 6$  are slightly degraded by finite-difference errors at the boundaries.

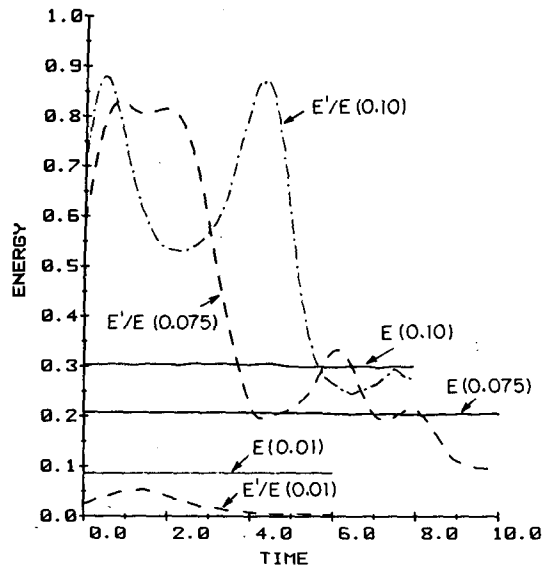


FIG. 4. The total energy  $E$  (solid lines) and the fractional perturbation energy  $E'/E$  (broken lines) as functions of time for  $A_0 = 0.01$ , 0.075, and 0.1.

as suggested by the linear results of Farrell (1982). However, as the wave continues to evolve, it develops a phase tilt to the right, thereby allowing energy to be restored to the mean flow as is evident in Fig. 4. Subsequently, the wave decays and leads to a "reverse S"  $\bar{u}$  profile with an inflection point which violates the Rayleigh and Fjørtoft sufficient conditions for stability. Thus, by this process the perturbation has produced a potentially unstable flow, even though the initial Couette profile is stable to normal-mode growths.

For  $A_0 = 0.1$  the solution for  $\bar{u}$  is similar to that shown in Fig. 3, but has larger amplitudes. At the smaller value of  $A_0 = 0.01$ , the solution behaves essentially linearly, with the mean flow remaining nearly

unchanged and the perturbation continually sheared by the mean flow for as long as the solution could be resolved.

Figure 4 shows global averages of the total energy  $E$  and the fractional perturbation energy  $E'/E$  as functions of time for  $A_0 = 0.01$ , 0.075 and 0.1. In each case the perturbation first exchanges energy with the mean flow but eventually decays. Ultimately, the solution becomes unstable, and the integration is stopped before instabilities become prominent. Since the total energy in these small-amplitude cases is much less than the minimum energy required by the gravest stationary mode [which can be evaluated from (9) to be  $E_{\min} = 0.86$ ], it is not surprising that the initial perturbation cannot reach the analytic  $\bar{u}$  profile in Fig. 3. Note, however, that since the perturbation eventually decays, the resulting mean flow always gains energy.

#### b. Nearly steady finite-amplitude solutions

In contrast to the above results, the solution for  $A_0 \geq 0.15$  evolves into a quasi-steady state which persists about the stationary solution found in Section 2.

In the case of  $A_0 = 0.15$ , the initial streamfunction (Fig. 5) quickly evolves into the quasi-steady pattern shown in Fig. 6 which closely resembles the analytic solution given in Fig. 1, although the initial-value solution also includes some contribution from  $l = 2$ , as is evident in Fig. 6b.

The time evolution of  $\bar{u}(y, t)$  is plotted for  $0 \leq t \leq 3.6$  in Fig. 7, which shows the solution effectively reaching a quasi-steady state by  $t \approx 1$ . Since in this case  $E/E_{\min} = 0.67 < 1$ , the results do not quite reach the analytic solution. The streamfunction pattern shown in Fig. 6 persists for about 4 time units beyond  $t \approx 1$  before becoming slowly degraded by small-scale instabilities at  $t \approx 6$ , where the integration is stopped. The relevant energy plots are given in Fig. 8. The sig-

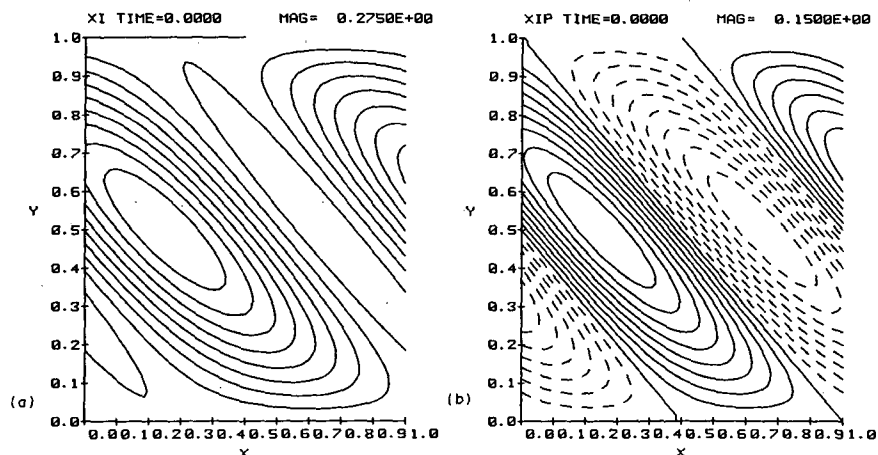


FIG. 5. The initial streamfunction for  $A_0 = 0.15$ . (a) The total streamfunction  $\psi$ ; (b) the perturbation streamfunction  $\psi'$ . Contour system as in Fig. 1.

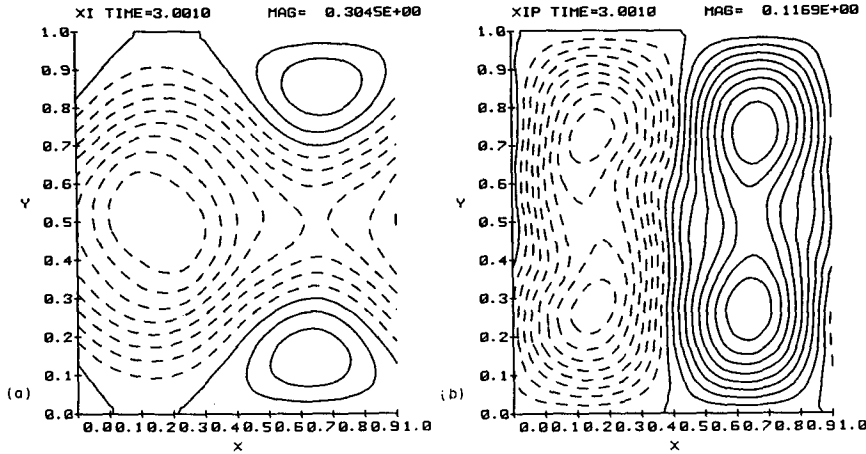


FIG. 6. The equilibrated streamfunction pattern for the case of  $A_0 = 0.15$ . (a) The total streamfunction  $\psi$ ; (b) the perturbation streamfunction  $\psi'$ . Contour system as in Fig. 1.

nificant results from this calculation are that (i) for a sufficiently large value of  $E$  (even if  $E/E_{min} < 1$ ), the initial-value solution can still achieve a quasi-steady state close to the gravest stationary mode, and (ii) the perturbation retains a substantial fraction of its initial energy rather than depositing it *in toto* into the mean flow, which would bring  $\bar{u}$  even closer to the analytic curve.

At the larger value of  $A_0 = 0.2$ ,  $E$  exceeds  $E_{min}$  by 10%, which makes it possible for  $\bar{u}$  to arrive at a quasi-steady state about the analytic solution, as shown in Fig. 9. Figure 9a shows that as the flow evolves, the solution briefly pauses in the neighborhood of  $t \approx 1.2$  before attaining the final equilibration in which  $\bar{u}$  virtually coincides with the analytic gravest mode (Fig.

9b). In terms of the total streamfunction, the solution pauses from  $t \approx 1.0$  to 1.5 about the finite-amplitude wave pattern shown in Fig. 10a (similar to Figs. 1a and 6a), then as the perturbation continues to decay, approaches the equilibrated pattern shown in Fig. 10b, which corresponds to the stationary solution (8) with a small wave amplitude. Figure 8 shows that, contrary to the  $A_0 = 0.15$  ( $E/E_{min} = 0.67$ ) case, here the wave transfers most of its energy to the mean flow and retains only a fraction of the "surplus" energy over  $E_{min}$ .

At the even larger value of  $A_0 = 0.25$ ,  $E/E_{min} = 1.7$ , in which case  $\bar{u}$  again equilibrates to the gravest mode,

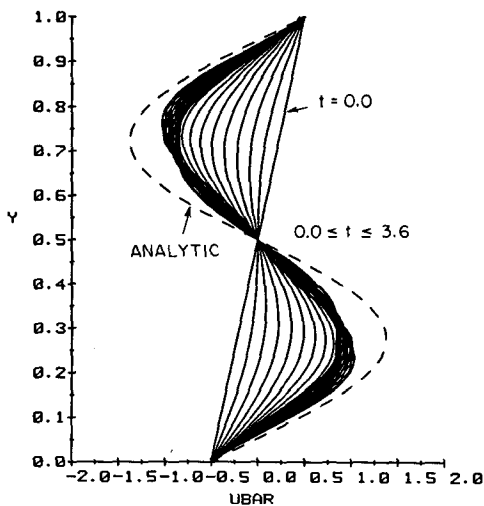


FIG. 7. The equilibration of  $\bar{u}$  for the case  $A_0 = 0.15$ . The interval of the plots is 0.1 nondimensional time units, and the total period covers  $0 \leq t \leq 3.6$ . The broken line is the analytic solution for the gravest stationary mode given by (10).

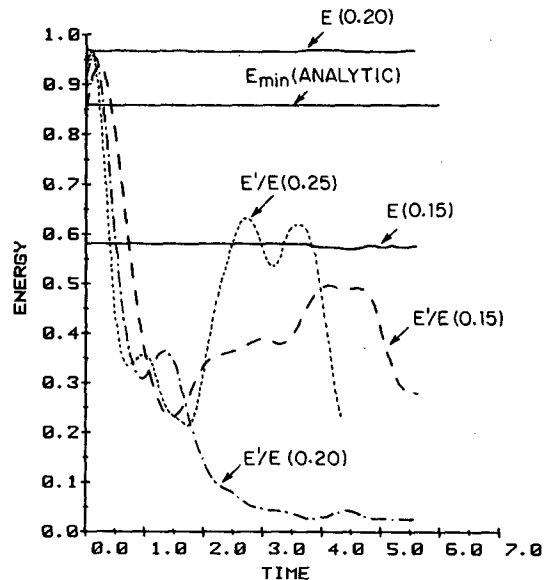


FIG. 8. The total energy  $E$  (solid lines) and the fractional perturbation energy  $E'/E$  (broken lines) as functions of time. The values of  $A_0$  are given in parentheses. The minimum energy  $E_{min}$  is evaluated from (9).

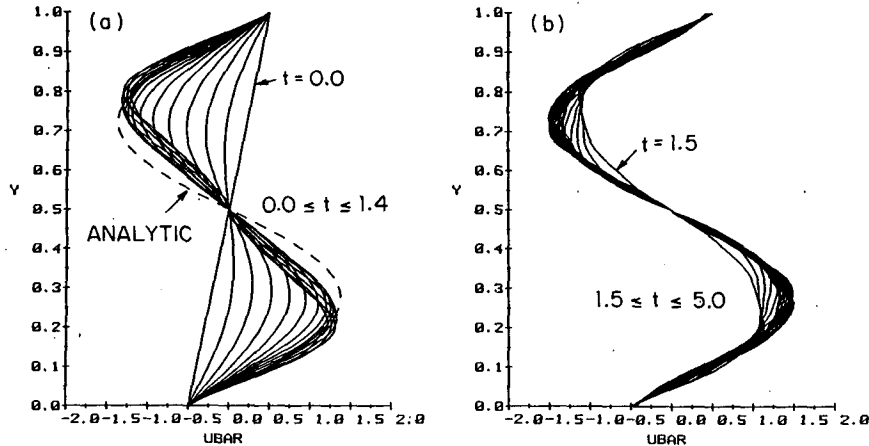


FIG. 9. The equilibration of  $\bar{u}$  for the case  $A'_0 = 0.2$ . (a) The initial time sequence for  $0 \leq t \leq 1.4$ ; the solution is shown to "pause" after  $t \approx 1.0$ ; the broken line indicates the exact analytic solution for the gravest stationary mode; (b) the final equilibration to the exact analytic solution. The interval of the plots is 0.1 nondimensional time units.

but the wave retains sufficient amplitude to produce a quasi-steady streamfunction pattern similar to Figs. 1a and 6a.

Common to these equilibrated solutions is that the equilibrated mean flow always has more energy than the initial profile, as can be deduced from the  $E'/E$  plots in Fig. 8.

As noted in Section 2, the vorticity gradient of the exact stationary solution is zero at all three critical levels; consequently there is no critical-layer instability. Fig. 11a displays the initial time sequence of the mean vorticity, which exhibits no singular behavior as the flow evolves in the presence of transients; Fig. 11b shows that as the flow equilibrates the mean vorticity gradient effectively vanishes at the critical levels, and

the calculated mean profile coincides with the analytic solution. Thus, in this nonlinear system in which the mean flow responds to wave transients, critical-layer instability is automatically eliminated. Similar conclusions have been reached by Geisler and Dickinson (1974) and Béland (1976) for forced Rossby waves, and Schoeberl and Lindzen (1984) for small barotropic disturbances. However, in our example the perturbation is of a sufficiently large amplitude so that the solution is not a local response to critical-layer dynamics but corresponds to the case discussed in Geisler and Dickinson where the mean-flow response time is much shorter than the critical-layer formation time.

Figure 12 plots the zonal Fourier coefficients for selected times for the case  $A'_0 = 0.2$ . As the flow evolves,

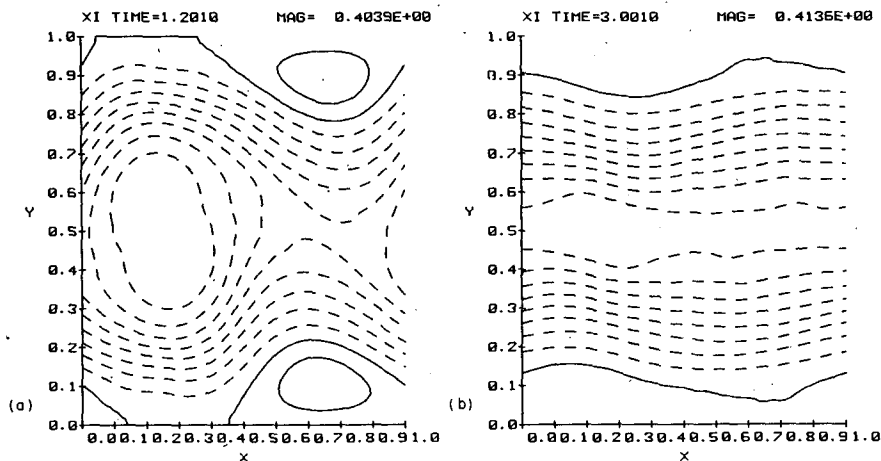


FIG. 10. The evolution of total streamfunction in the case of  $A'_0 = 0.2$ . (a) The total streamfunction  $\psi$  at  $t = 1.2$ , which coincides with the brief "pause" in Fig. 9a; (b) the long-time solution which consists of a small perturbation superimposed on the mean. The contour system is the same as that for Fig. 1.

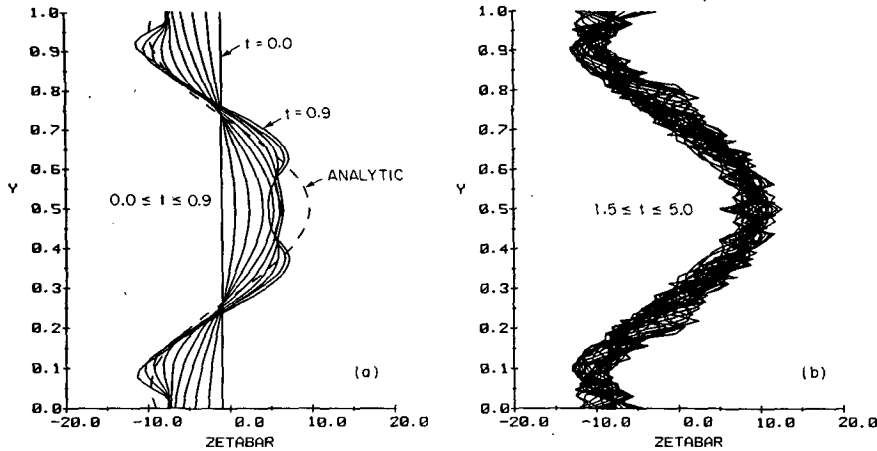


FIG. 11. The evolution of the mean vorticity in the case of  $A_0 = 0.2$ . (a) The initial time sequence for  $0 \leq t \leq 0.9$  (with intervals of 0.1 nondimensional time units), which shows the emergence of small-scale waves in the cross-channel direction; (b) the equilibrated profile, which coincides with the analytic stationary solution given by (8). Since the stationary solution may be unstable to small-scale disturbances, as discussed in section 2, some slowly-amplifying noise is evident in the numerical results.

the energy initially increases in both the mean flow and the small scales, before gradually filling up the entire spectrum. Ultimately, the flow is unstable to small-scale disturbances, as suggested by the solution at  $t = 5.6$ . The nature of this instability may be physical or numerical or both; detailed stability properties will be examined in a subsequent study. The main point of this study is to demonstrate that a quasi-steady solution could actually persist for long periods before being destroyed by small-scale instabilities.

To ensure that the foregoing solution does not depend on the model resolution, we have performed calculations with half of the standard resolution and ob-

tained the same result. Thus, the emergence of the nearly steady state does not rely on waves smaller than  $k = 16$ . However, the equilibration process does involve zonal waves shorter than  $k = 1$ . Using a severely truncated model with only  $k = 0$  and 1, we found that the solution does not reach a quasi-steady state for either  $A_0 = 0.15$  or 0.2. The continuous spectrum of waves which form the transients must play an essential role in the equilibration process; this is emphasized by Béland (1976) and Killworth and McIntyre (1985).

*c. Other phase orientations*

In experiments where the initial wave is vertical or has a phase tilt to the right (with the mean-flow shear), the mean-flow acceleration it induces is in the opposite direction from that for approaching the stationary state; the result is that the wave decays monotonically with time and leads to a "reverse S"  $\bar{u}$  profile, as anticipated by the linear study of Farrell (1982).

**4. Conclusions**

While this study is preliminary, the physical process outlined here addresses a number of important questions. A finite-amplitude wave can interact with a mean flow to produce a stationary solution. This points to a possible mechanism for exciting isolated nonlinear features, making them accessible from a range of initial conditions. The results also show that in an evolving nonlinear flow, the development of a critical level, which is known to play a crucial role in linear, quasi-linear, and weakly nonlinear wave dynamics, is accompanied by the neutralization of the local mean vorticity gradient, thus allowing the solution to achieve a smooth modal structure.

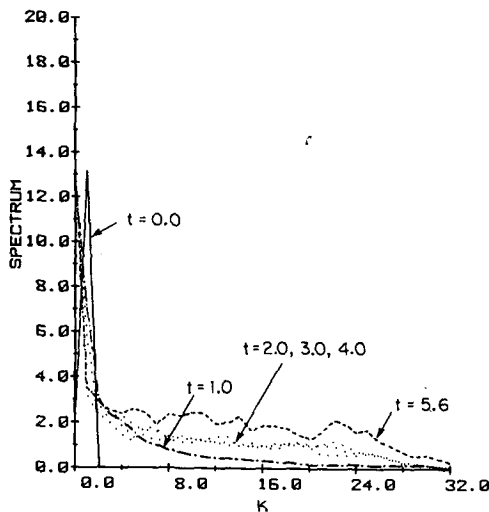


FIG. 12. The zonal Fourier coefficients at selected times in the case of  $A_0 = 0.2$ . The curves are as marked, with the dots corresponding to the nearly-steady results at  $t = 2, 3$  and 4.



The major results of this study are

1) A finite-amplitude barotropic wave in an inviscid Couette flow can evolve into a shape-preserving, nearly stationary solution, providing that its initial phase tilt is in the opposite orientation from that of the mean-flow shear.

2) For sufficiently large initial wave amplitudes, the initial-value solution tends asymptotically to and persists about the stationary state predicted by our exact, analytic solution; this requires that the total energy of the initial state be comparable to or greater than that of the analytic stationary mode. For small initial amplitudes, the wave does not equilibrate and is continually sheared as predicted by the linear theory.

3) As the flow equilibrates, critical levels emerge in the flow but the mean vorticity gradient vanishes at these levels, so that Bretherton's condition for the existence of nonsingular neutral modes is satisfied.

4) The parameter  $\beta$  enters only through the phase speed, and is not central to the physics of the stationary mode.

5) The equilibrated mean-flow profile does not satisfy Arnol'd's sufficient condition for stability, and extended integrations show that the solution eventually breaks down in the presence of small perturbations.

The purpose of this article is to offer a simple demonstration of the connection between the initial-value and nonlinear stationary solutions under a range of initial conditions. It is hoped that this example will help promote further studies of the role of the continuous spectrum of transients and its relation to the equilibration process.

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