

Optimal Excitation of Neutral Rossby Waves

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ABSTRACT

Properly configured disturbances are known to be effective in transferring the kinetic energy of a mean shear flow to neutrally stable modal and nonmodal waves. Consideration of perturbation energetics requires that such disturbances produce down-gradient momentum fluxes which are associated with perturbation phase lines oriented against the mean shear. Initial conditions chosen arbitrarily, except that they satisfy this requirement, have been shown to result in robust excitation of neutral waves. A question naturally arising from such studies is whether there exists, in some well-defined sense, a best or most effective choice of initial conditions which optimally excites the waves. This question is addressed as a variational problem, and examples of optimal initial conditions are identified for the barotropic β -plane channel. These examples include the most effective excitation of a given neutral Rossby mode and the most rapidly growing perturbation for a given time period without restriction on spectral composition.

1. Introduction

Baroclinic and barotropic instability are regularly invoked to explain the existence of traveling waves in the atmosphere. While it is sometimes possible to find an inviscid linear mode which is both unstable and plausibly related in structure to an observed wave, this explanation fails in the case of the neutral retrograde Rossby modes of planetary scale, examples of which include the so-called 5 and 16 day waves (Deland, 1964; Eliassen and Machenhauer, 1965; Lindzen et al., 1984; Ahlquist, 1985). Observational and theoretical studies of these waves have been reviewed by Madden (1979) and Salby (1984); they are the neutral global Rossby modes modified by background wind shear and sphericity effects. In light of their neutrality, it is often assumed in linear studies that these disturbances are resonantly excited by a periodic or stochastic forcing (Hirota, 1971; Madden, 1975; Garcia and Geisler, 1981). Alternatively, inclusion of nonlinearity makes it possible that they are driven by the inverse cascade of two-dimensional (2-d) turbulence theory as these planetary modes are the natural repository for energy injected at the synoptic scale and transferred upscale under the assumption that boundary interactions can be ignored (Charney, 1971).

Planetary-scale Rossby waves are often poorly predicted by models, and this deficiency contributes to both forecast and climatology errors (Daley et al., 1981; Roads and Somerville, 1982). In light of this, it is of interest to study further the physical processes involved in the excitation of these neutral modes.

Although the problem is posed here in terms of the barotropic β -plane channel model in order to provide a clear focus for the discussion to follow, the results apply as well to the baroclinic problem with a suitable redefinition of variables and boundary conditions. Neither of these modifications alters the qualitative conclusions obtained in the barotropic model. For examples of baroclinic neutral modes excited by energy-releasing but nonoptimal initial perturbations, see Farrell (1982).

There is no inherent scale selection in the mechanism to be discussed such as is imposed by the maximum growing mode in exponential instability, and it pertains equally to the excitation of waves of synoptic or even subsynoptic scale. The planetary Rossby wave/barotropic β -plane channel was chosen as an illustrative couple because of its simplicity and relation to familiar ideas of Rossby wave dispersion.

The notion of an optimal perturbation requires a norm. In order to explore the sensitivity of results to different norms, two representative choices are made: the L_2 and energy norms. In addition to the freedom of choosing the norm, there is the additional choice of constraints on the optimization. In order to explore the sensitivity of results to constraints, two examples are solved: a maximization of a chosen mode in section 3, and a maximization over a chosen interval of time with no restriction on the spectral composition in section 4.

2. The barotropic model

The barotropic vorticity equation and boundary conditions governing the evolution in a β -plane channel of Rossby waves impressed on a zonal flow $U(y)$ is

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$$\left[\frac{\partial}{\partial t} + U(y) \frac{\partial}{\partial x} \right] \nabla^2 \psi + (\beta - U_{yy}) \frac{\partial \psi}{\partial x} = 0, \tag{1}$$

$$\psi = 0 \quad (y = y_1, y_2)$$

where

$U(y)$ basic state zonal velocity
 ψ perturbation stream function
 β northward gradient of the Coriolis parameter.

With the assumption of a constant shear, $U(y) = Sy$, this equation may be nondimensionalized using the half channel width $L/2$ as a length scale and the inverse shear S^{-1} as a time scale:

$$\left[\frac{\partial}{\partial t} + y \frac{\partial}{\partial x} \right] \nabla^2 \psi + \beta \frac{\partial \psi}{\partial x} = 0 \tag{2a}$$

$$\psi = 0 \quad (y = -1, +1) \tag{2b}$$

in which variables are regarded as nondimensional and specifically

$$\beta = \frac{\beta_{\text{dim}} L}{S} \tag{3}$$

Assuming a solution of the form $\psi = \psi(y, t)e^{ikx}$ gives

$$\left(\frac{\partial}{\partial t} + yik \right) \left(\frac{\partial^2}{\partial y^2} - k^2 \right) \psi + \beta ik \psi = 0. \tag{4}$$

Assuming further that $\psi(y, t) = \psi(y)e^{-ickt}$ results in the eigenvalue problem:

$$\psi_{yy} + \left(\frac{\beta}{y-c} - k^2 \right) \psi = 0 \tag{5a}$$

$$\psi = 0 \quad (y = -1, +1). \tag{5b}$$

The first limit of interest is $S \rightarrow 0$ which yields by inspection of the nondimensionalization the familiar Rossby wave in the absence of basic flow:

$$\psi_{yy} - \left(k^2 + \frac{\beta}{c} \right) \psi = 0 \tag{6a}$$

$$\psi = 0 \quad (y = -1, 1) \tag{6b}$$

with eigenfunctions

$$\psi_n = \sin \frac{n\pi}{2} (1 + y) \quad (n = 1, 2, \dots) \tag{7a}$$

and eigenvalues

$$c_n = \frac{-\beta}{k^2 + (n\pi/2)^2}. \tag{7b}$$

The normal modes of the self-adjoint operator in (6) form a complete, orthogonal set of functions satisfying the boundary conditions. The structure in y of an initial condition which satisfies requirements for the existence of a Fourier series representation can be written as

$$\psi(y, t = 0) = \sum_{n=1}^{\infty} \alpha_n \sin \left(\frac{n\pi}{2} \right) (1 + y)$$

$$\alpha_n = \int_{-1}^{+1} \psi(y, t = 0) \sin \left(\frac{n\pi}{2} \right) (1 + y) dy.$$

The subsequent evolution at a fixed zonal wavenumber k of orthogonal modes (7a) each associated with its phase speed (7b) is given by

$$\psi(y, t) = \sum_{n=1}^{\infty} \alpha_n e^{-ikc_n t} \sin \left(\frac{n\pi}{2} \right) (1 + y).$$

The real part of $\psi(x, y, t) = e^{ikx} \psi(y, t)$ is identified with the physical solution. The energy integrated over the domain with a bar denoting the x average is

$$\begin{aligned} \langle \bar{E} \rangle &= \int_{-1}^{+1} \frac{|\bar{\psi}_x|^2 + |\bar{\psi}_y|^2}{2} dy \\ &= \int_{-1}^{+1} \frac{k^2 \psi^*(y) \psi(y) + \psi_y^*(y) \psi_y(y)}{4} dy. \end{aligned}$$

The orthogonality of the modes permits this to be expressed as

$$\langle \bar{E} \rangle = \sum_{n=1}^{\infty} \frac{\alpha_n^2}{4} (k^2 \langle \psi_n^* \psi_n \rangle + \langle \psi_{n,y}^* \psi_{n,y} \rangle) \tag{8}$$

$$= \sum_{n=1}^{\infty} \frac{\alpha_n^2}{4} \left[k^2 + \left(\frac{n\pi}{2} \right)^2 \right]. \tag{9}$$

There is no time dependence of the total energy, even though the streamfunction is time varying. The energy is partitioned among the modes so that the total is given by a sum over the individual mode energies.

The second limit of interest results from taking β large, corresponding by (3) to a small shear S or a large channel L ; for reference, typical midlatitude synoptic scale values of $\beta_{\text{dim}} = 1.62 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$, $L = 6 \times 10^6 \text{ m}$, and $S = 5 \times 10^{-6} \text{ s}^{-1}$ imply $\beta = 19.4$.

The eigenproblem (5) is now singular for phase speeds $-1 < c < 1$ and the only nonsingular normal modes have phase speeds outside this range; specifically they are retrograde modified Rossby waves with $c < -1$. Examination of the dispersion relation without shear (7b) suggests that the infinity of modes collecting around $c = 0$ will be lost no matter how small a nonzero shear is imposed, thus destroying the completeness of the normal mode spectrum. Less obviously but of equal dynamical importance is the loss of orthogonality between the normal modes themselves and between the normal modes and the complement continuous spectrum of singular modes. This destroys the Parseval's relation (8) and allows the total energy of a composite disturbance to vary even in the absence, as here, of exponentially growing normal modes. A historically interesting discussion of this distinction and its dynamical significance is given by Orr (1907).

It is useful to predict the modification of Rossby wave eigenvalues and eigenfunctions produced by an imposed shear through an expansion of (5) in powers of β^{-1} (Drazin et al., 1982. See Appendix A). Phase speed is relatively insensitive to the imposition of shear,

TABLE 1. Comparison between the calculated Rossby mode phase velocity correct to first order and the numerically derived value for the choice

$$\beta = 20, \quad k = \frac{\pi}{2}.$$

Mode	Calculated	Actual
1	-4.05	-4.11
2	-1.62	-1.76
3	-0.81	-1.10

being modified at second order while the structure is changed at first order and the correction increases as n^3 for higher order modes. In Table 1 corrected eigenvalues are compared to exact numerical values. Here a flow with $\beta = 2.00$, which we have seen corresponds to synoptic scale nondimensionalization, leaves only three normal modes with $c < -1$ while the perturbation results predict two. It is quite generally found in examples with physically realistic shear that only a small number of normal modes remain, and most disturbances are supported by the continuous spectrum (Dikiy and Katayev, 1971; Kasahara, 1980). Modification to the structure of the remaining normal mode eigenfunctions is equally important, as it is the reason for the dynamically significant loss of orthogonality between the modes. First-order correction is found to be accurate for the gravest mode, but errors increase as the structure becomes highly modified near the cutoff phase speed for model solutions. For this reason, it is best to proceed by solving the equation numerically.

The equation of motion (4) and boundary conditions in general operator form are

$$\frac{\partial \psi}{\partial t} + iL\psi = 0 \quad (10)$$

$$L \equiv k\nabla^{-2}(\gamma\nabla^2 + \beta). \quad (11)$$

Solutions of this problem in finite difference form with the operator expressed as a matrix are the discrete analogues of both the normal modes and the continuous spectrum (Yanai and Nitta, 1968):

$$LE_j = \sigma_j E_j \quad (12)$$

$$\psi_j(t) = E_j e^{-i\sigma_j t}. \quad (13)$$

The E_j, σ_j pairs are the eigenvectors and eigenvalues of the matrix L . An initial perturbation evolves according to

$$\psi(t) = \sum_j \alpha_j E_j e^{-i\sigma_j t}. \quad (14)$$

With the α_j chosen as the projection of $\psi(0)$ on the modes,

$$\alpha = E^{-1}\psi(0),$$

E being the matrix whose columns are the eigenvectors. The self-adjoint operator corresponding to the Rossby wave with no mean flow shear has as counterpart a symmetric matrix with real eigenvalues corresponding

to real phase speeds and a complete set of orthogonal eigenvectors corresponding to the orthogonal modes. With the addition of shear, the self-adjointness of the operator is lost and its matrix counterpart becomes nonsymmetric; the modes are no longer orthogonal and can interact with each other energetically, a fixed energy can no longer be associated with a given mode, and the Parseval's relation does not hold. This interaction allows the energy of the mean flow to be tapped by stable modes to excite and maintain a neutral wave field. For alternative interpretation of these ideas see Held (1985) and North (1984).

The choice of an initial perturbation which is effective in utilizing the mean flow energy to increase the wave amplitude and energy is guided by considerations of energy integral relations (Pedlosky, 1987; Farrell, 1987). These show that a favorable perturbation has phase lines oriented opposite the shear, producing a down-gradient Reynolds stress. Guided by this consideration, perturbations were chosen in the form of plane waves or plane waves modulated by localizing envelope functions (Farrell, 1982, 1984, 1985, 1987), and in all cases with neutral normal modes a robust transfer of mean flow energy to the normal modes was found. In light of these results, the question naturally arises: is there a best choice of initial perturbation which will most efficiently excite a chosen neutral mode?

3. The optimization problem for neutral modes

The concept of an optimal function requires that a norm be chosen. Historically and for reasons of analytical simplicity, the square of the magnitude or L_2 norm comes first to mind. In geostrophic flow it is proportional to the integrated square of the perturbation pressure. Perhaps a more physically appropriate measure of a perturbation is its integrated energy, proportional to the integrated square of the velocity amplitude. We begin with the L_2 norm.

The functional to be minimized expressed in vector form is $\psi^* \cdot \psi$ where the star indicates the Hermitian transpose. With the use of (14) at $t = 0$:

$$(E\alpha)^* \cdot (E\alpha) = \alpha^* E^* E \alpha \equiv \alpha^* A \alpha; \quad (15)$$

A is the matrix of the positive definite quadratic form associating with each spectrum α the square of the L_2 norm of the streamfunction ψ which results from the modal sum (14).

Choosing as a constraint that the i th mode be of unit magnitude, the variational problem is to render stationary the functional

$$F = \alpha^* A \alpha + \lambda(\alpha \cdot \xi_i - 1)$$

where ξ_i is the unit column vector. Setting the first variation in α to zero gives

$$A^T \alpha = -\lambda \xi_i. \quad (16)$$

The optimal spectrum is the solution recognizing A to be symmetric:

$$\alpha = -\lambda A^{-1} \xi_i.$$

The solution is completed by choosing the Lagrange multiplier λ so that $\alpha_i = 1$.

This result is related to the adjoint eigenfunctions through the concept of biorthogonality. Recall that the matrix corresponding to (11) is Hermitian, in fact, real symmetric, implying it has orthogonal eigenvectors only in the unshered case. However, in the sheared flow, the general matrix has a companion adjoint matrix with eigenvalues which are the complex conjugates of the eigenvalues of the original matrix. The companion eigenvector to the conjugate eigenvalue is orthogonal to all eigenvectors of the original matrix except its companion (Noble, 1969). From (15) and (16), the L_2 optimal initial condition for exciting a given mode is clearly the adjoint mode. This relation can also be shown to hold between a time invariant spatial pattern of excitation and the driven stationary solutions (Branstator, 1985).

Taking advantage of the observation that the desired initial condition is the eigenvector of the adjoint matrix, the optimization problem is solved for all modes simultaneously by eigenanalysis of the adjoint. Figure 1 shows the three modes for $\beta = 20$ and $k = \pi/2$, with the corresponding optimal perturbations. It is immediately apparent from an examination of Fig. 1 that the gravest eigenmode and its adjoint have a similar

structure. This implies that this mode is nearly orthogonal to the remaining spectrum and that the energy of the mean flow is only weakly available to excite this mode. Direct calculation shows a growth in L_2 norm of approximately 20% when the optimal initial condition is used. The remaining normal modes have progressively greater differences between the mode and its optimal excitation as n increases and the cutoff phase speed is approached, which implies a greater potential for extracting energy from the mean flow. The development of the mode 3 optimal initial condition is shown in Fig. 2. In this figure and those to follow, the normalized square of the L_2 norm, and maximum streamfunction amplitude are indicated at each time so that the resolution of the plot need not be compromised by a constant contour interval.

Notice the concentration of the optimal initial perturbation near $y = -1$ where the velocity has its easterly maximum and the formation of an energy-releasing tilt as the perturbation grows. The time scale for the development is approximately 10 nondimensional units, each of which represents the time of advection across the half-channel width by the maximum mean flow velocity.

A second square norm of interest is energy defined as

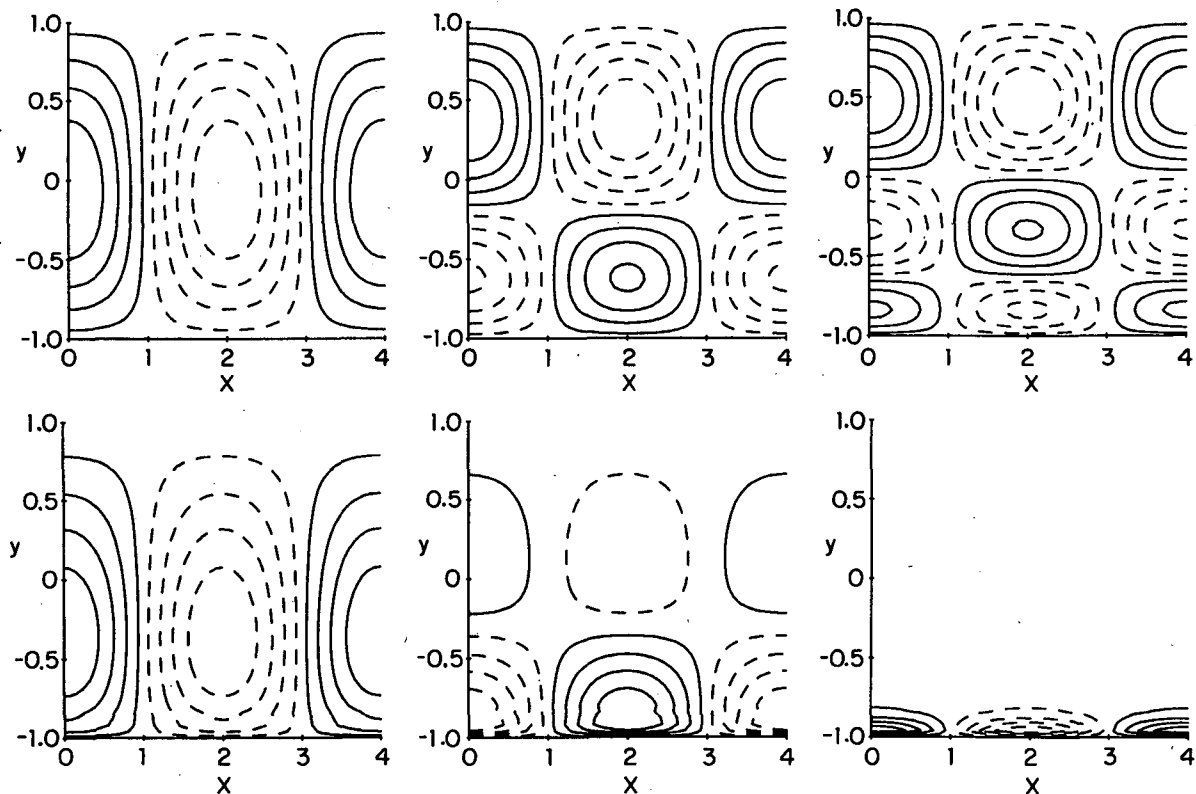


FIG. 1. The neutral retrograde Rossby modes for $\beta = 20$, $k = \pi/2$. In the top panels are the modes themselves and in the bottom are their adjoint companion modes which are the L_2 norm optimal initial perturbations.

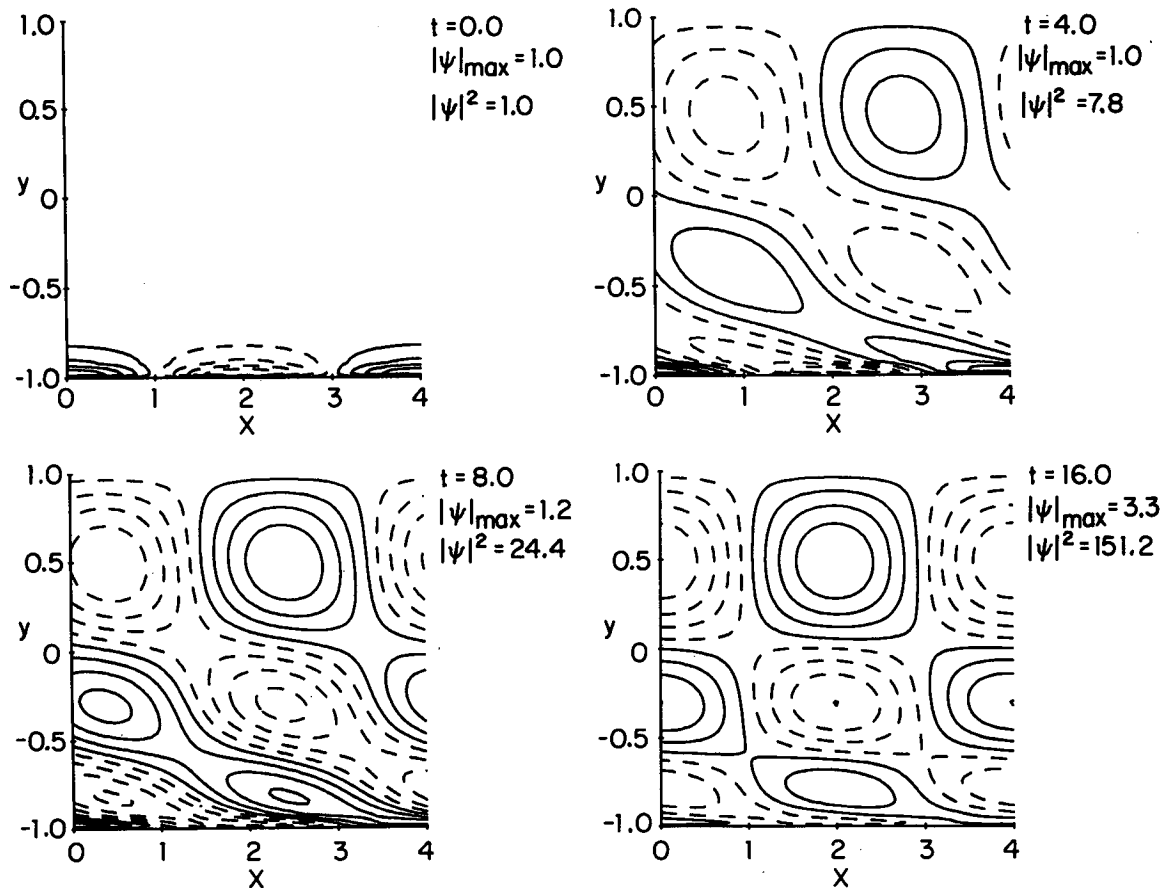


FIG. 2. Time development of the optimal initial perturbation for mode 3 in the L_2 norm with $\beta = 20$ and $k = \pi/2$.

$$4\bar{K} = k^2\psi^* \cdot \psi + \psi_y^* \cdot \psi_y.$$

This norm can be expressed in terms of the spectrum α as

$$4\bar{K} = \alpha^* \mathbf{B} \alpha$$

$$\mathbf{B} \equiv k^2 \mathbf{A} + \mathbf{E}_y^T \mathbf{E}_y,$$

where \mathbf{E}_y is the matrix of column eigenvector derivatives.

The optimization proceeds as before with solution

$$\alpha = -\lambda \mathbf{B}^{-1} \epsilon_i$$

with λ chosen to make the projection on the i th mode unity.

The minimum energy initial condition for mode 3 is shown in Fig. 3 along with its time evolution. The energy growth rate for this example (Fig. 4) can be compared to a typical barotropic instability growth rate as in Gill (1982, p. 567) which is $(1/E)dE/dt = 0.402$, recalling, of course, that there are no instabilities in our model. Notice also that the energy norm which in effect penalizes derivatives in contrast to the L_2 norm results in a smoother optimal perturbation.

4. The optimal growth problem

Because of the importance of neutral modes in the atmosphere and in GCM simulations, we have concentrated on their excitation in section 3. However, the method employed there can be applied to another problem of interest which results from relaxing the constraint that a given mode have a specified amplitude. The question is to find the perturbation for a given norm which produces the maximum growth in a fixed time interval. The motivation here is physical in that growth rate is often of more interest in initial value problems than is maximum total growth. While it may be of theoretical interest to know that arbitrarily large growth can be obtained in the free shear or the Couette problem if the perturbation be properly chosen, even so such growth is not likely to be of physical significance if it results from a prolonged period of small growth. This notion is made concrete by choosing as physically relevant time scales of the order of a few advection periods of the channel width at the maximum mean flow velocity and searching for explosively developing perturbations.

Choosing first the L_2 norm, we seek the perturbation

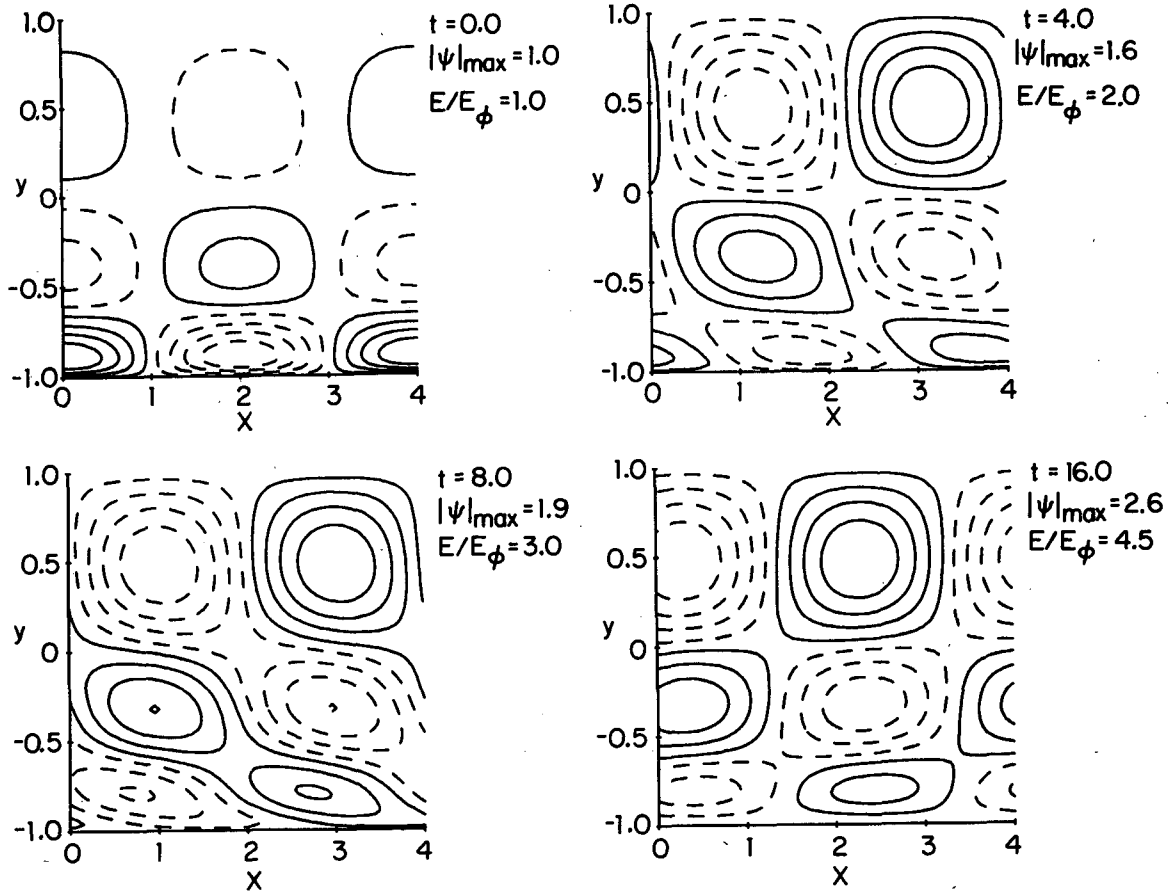


FIG. 3. As in Fig. 2 except for the energy norm.

of unit initial norm which obtains the maximum norm in the specified time.

The functional to be rendered stationary is

$$F = \alpha^* A_t \alpha + \lambda (\alpha^* A_0 \alpha - 1),$$

$$A_t \equiv E_t^T E_t,$$

where E_t is the matrix of column eigenvectors each advanced by its appropriate phase in the time interval according to (13). Also, the transpose is understood to be Hermitian as the matrix E is no longer real.

The requirement of stationarity is

$$A_t \alpha + \lambda A_0 \alpha = 0$$

$$(A_0^{-1} A_t + \lambda I) \alpha = 0.$$

This is an eigenproblem with the Lagrange multiplier serving as the eigenvalue. There are n candidate stationary spectra, one of which is the desired optimum.

Choosing a nondimensional time interval $t = 8$ results in the optimal perturbation which is shown in Fig. 5 together with its development at subsequent times. Analogous results for the energy norm are in Fig. 6. The energy growth rate for this example is shown in Fig. 7. The similarity of the channel optimal initial perturbations in Fig. 5 and Fig. 6 is not too surprising in view of the fact that in constant free shear the am-

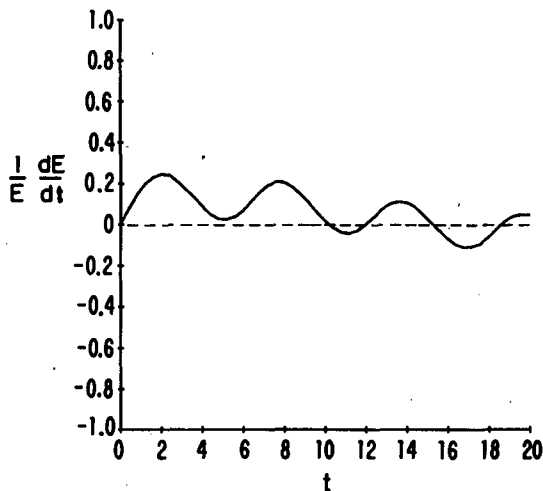


FIG. 4. The energy growth rate for the example in Fig. 3.

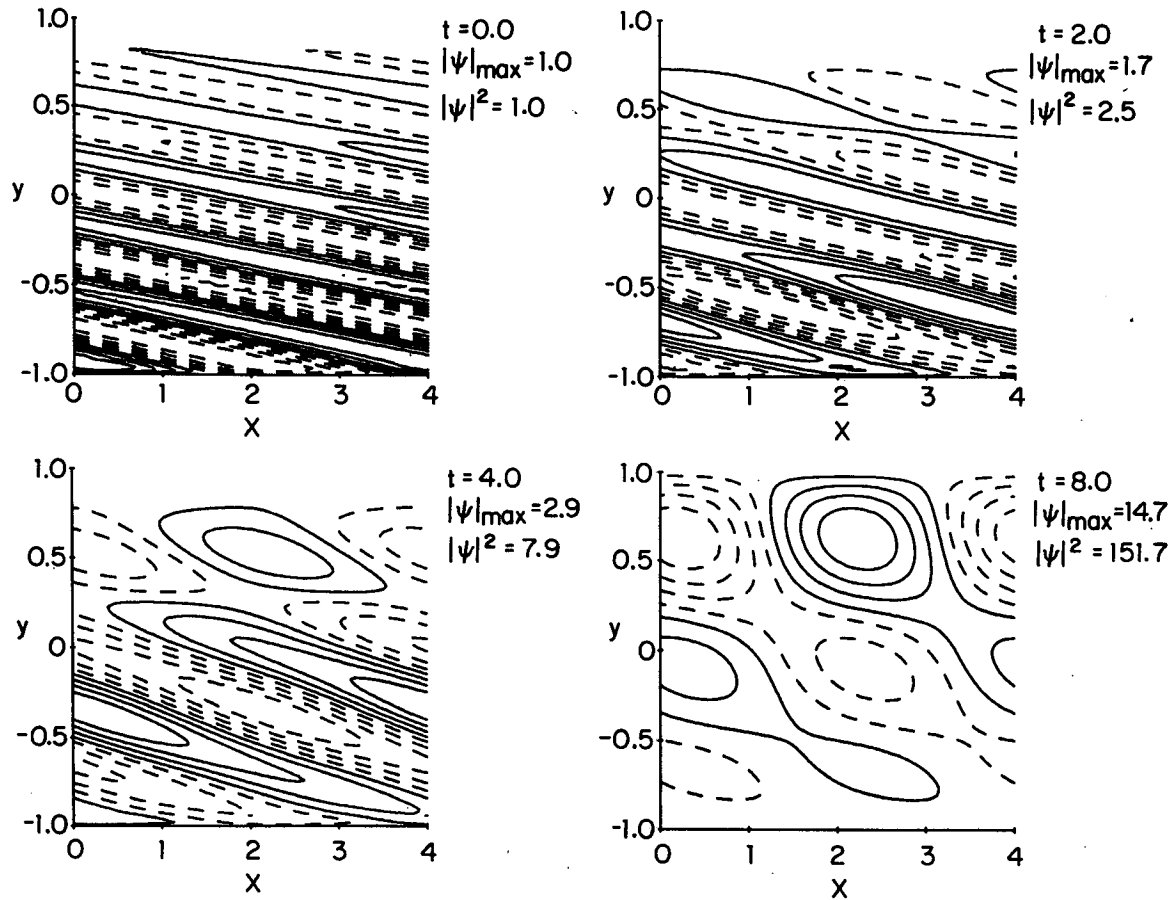


FIG. 5. Time development of the optimal growth rate perturbation for an interval $t = 8$ in the L_2 norm.

plitude and energy optimal perturbations are identical (Tung, 1983).

5. Discussion and conclusion

The primary store of atmospheric kinetic and potential energy is the zonal mean flow, and the growth of wavelike disturbances most often results from transfer of mean energy to the perturbation scale. While any shear flow possesses such a store of energy, it is generally held to be unavailable for transfer to the perturbation scale unless certain highly restrictive conditions associated with the existence of exponentially growing normal modes are satisfied. Moreover, even in the event that these requirements are met, the structure of the growing perturbation is predicted to be that of the maximum growth rate mode while observations show a wealth of perturbation scales and structures. The observed excitation of neutral modes places this issue in sharp focus: if these waves cannot extract energy directly from the mean flow, then they must be driven stochastically or by nonlinear transfer of energy from other, presumably unstable, scales. We have seen that this deduction is mistaken and that over time scales

associated with the observed growth of neutral waves a perturbation can be found which supplies energy to neutral modes at a rate comparable to that expected from the growth of a maximally unstable mode.

A favorable perturbation can be picked from general notions of energetics, i.e., it should have phase lines oriented opposite the mean shear so as to produce down-gradient Reynolds stresses. Still, it is of interest to find the optimal initial condition which in some well-defined sense most effectively transfers the energy of a mean flow to neutral disturbance. Here we have solved this problem for the simple model of Rossby waves in a constant shear β -plane channel. These results have application to the excitation of neutral planetary waves in the troposphere and also in the stratosphere where the explosive growth of planetary waves has been associated with the nonlinear breakdown of the mean flow (Hirooka and Hirota, 1985; Smith, 1985). In this connection, two further points can be made concerning barotropic vs baroclinic dynamics. Except for the boundary conditions which support additional modes in the baroclinic problem, the linear dynamics are essentially parallel in these two problems when a suitable identification of variables is made; in particular, the

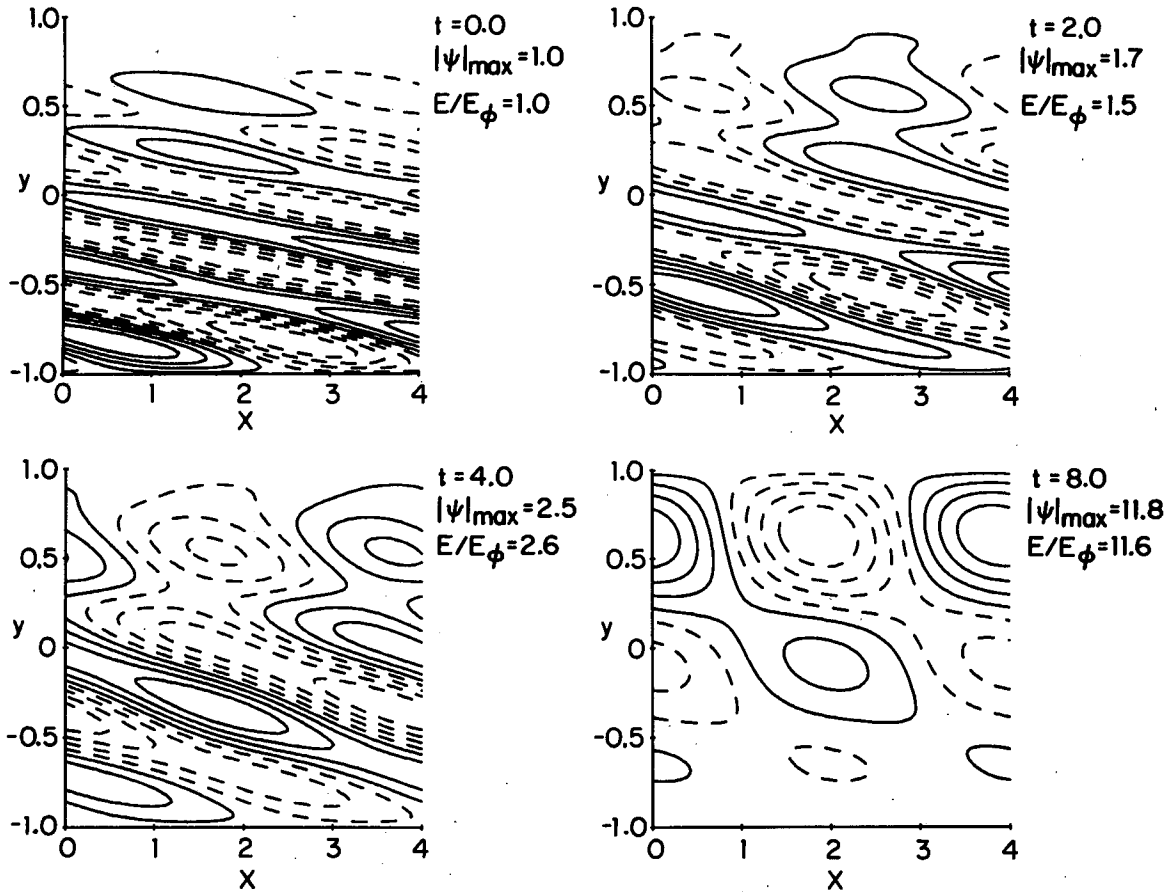


FIG. 6. As in 5 except for the energy norm.

zonal velocity perturbation in the barotropic problem is identified with the temperature perturbation in the baroclinic. It then follows that similar results to those obtained here could be found for the problem of op-

timal excitation of, say, the Charney problem or, more typically, of the polar night jet.

Viewing the optimal growth example as a pure baroclinic process, Fig. 7 outlines a scenario of growth by positive baroclinic conversion followed by a symmetric decay by negative baroclinic conversion. This results in a characteristic symmetric tent-shaped amplitude profile in time which can be interpreted as the "life cycle" of a baroclinic wave; such a phenomena has been observed by Randel et al. (1987) in the Southern Hemisphere stratosphere. A life cycle of growth by barotropic and/or baroclinic conversion followed by a partial or complete decay by the same processes acting in the reverse sense is a general result of linear initial value problems and does not require a nonlinear equilibration of the wave to intervene between the growth and decay phases as is the case for exponential normal modes (Simmons and Hoskins, 1978).

The nonlinear problem is more complex in the baroclinic case where the alteration of the mean flow by the growing wave is modified by induced secondary circulations. However, observations of stratospheric warmings (Smith, 1985) show a remarkable similarity to the equilibration scenario found by Hou and Farrell

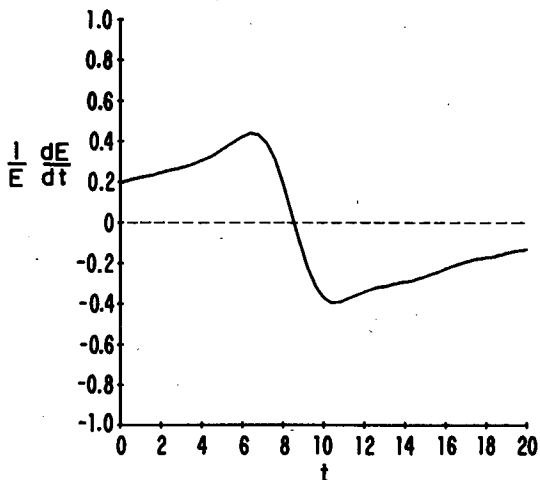


FIG. 7. The energy growth rate for the example in Fig. 6.

(1986) for the barotropic channel problem in which excitations of insufficient strength to exceed a transition amplitude were reversible, in qualitative similarity to the linear problem, but that perturbations of a sufficient amplitude caused the flow to make a discontinuous change to a nonlinear permanent form solution.

Returning to the optimal growth problem, a prediction which may help to distinguish the mechanism just outlined from other sources of excitation such as a nonlinear cascade is the fact that some modes are nearly orthogonal to the rest of the spectrum, for example, the gravest mode in Fig. 1. Such modes interact weakly with the other waves and do not efficiently draw upon the mean flow for their excitation. On the other hand, modes such as that in Fig. 2 would be expected to be more prevalent in observations if this mechanism is operating. This shorter time scale for growth and smaller amplitude maximum for low-order modes as compared to higher-order modes was found in the observations of Lindzen et al. (1984, their Fig. 3).

In summary, neutral waves can effectively draw on mean flow energy for their excitation given a favorable initial perturbation and, further, it is possible, given a norm, to find the best initial perturbation.

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APPENDIX A

Expansion of Linear Barotropic β -plane Channel Model Eigenrelations in Powers of β^{-1}

Substitution of the expansion

$$c_n = -\beta c_{n0} + c_{n1} + \beta^{-1} c_{n2} + \dots$$

$$\psi_n = \psi_{n0} + \beta^{-1} \psi_{n1} + \dots$$

in (5), results at order 0, in

$$\psi''_{n0} + \left(\frac{1}{c_{n0}} - k^2 \right) \psi_{n0} = 0$$

$$\psi = 0 \quad y = -1, +1$$

The solution is that of the unsheared problem:

$$\psi_{n0} = \sin \frac{n\pi}{2} (1 + y)$$

$$C_{n0} = \left[k^2 + \left(\frac{n\pi}{2} \right)^2 \right]^{-1}, \quad n = 1, 2, \dots$$

At order β^{-1} the equation is

$$\psi''_{n1} + \left(\frac{1}{c_{n0}} - k^2 \right) \psi_{n1} = \frac{(y - c_{n1})}{c_{n0}^2} \psi_{n0}$$

Solvability requires

$$c_{n1} = \int_{-1}^{+1} y \sin^2 \frac{n\pi}{2} (1 + y) dy = 0,$$

which vanishes due to the antisymmetry of the argument so that the first nonzero correction to the phase speed occurs at $O(\beta^{-2})$. Therefore, phase speed for the unsheared problem gives a good approximation to the phase speed for moderate shear.

The eigenfunction correction from

$$\psi''_{n1} + \left(\frac{n\pi}{2} \right)^2 \psi_{n1} = \left[k^2 + \left(\frac{n\pi}{2} \right)^2 \right]^2 y \sin \left(\frac{n\pi}{2} \right) (1 + y)$$

is found to be

$$\psi_{n1} = \frac{\left[k^2 + \left(\frac{n\pi}{2} \right)^2 \right]^2}{n\pi} \left\{ \left[(1 + y) - \frac{(1 + y)^2}{2} \right] \cos \frac{n\pi}{2} \times (1 + y) + \frac{(y - 1)}{n\pi} \sin \frac{n\pi}{2} (1 + y) \right\}.$$

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