

Developing Disturbances in Shear

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ABSTRACT

The origin and growth to moderate amplitude of disturbances in shear flow has been traditionally ascribed to the linear modal instability of the flow. Recent work on initial value problems has suggested that nonmodal growth of perturbations may be of equal and perhaps greater importance, at least in cases of rapid development. Examples of robust growth in model problems which support no instabilities and in baroclinic flows with realistic Ekman damping for which the exponential modes have zero or negative growth rates have been shown. Such examples have focused attention on the perturbations which are configured to tap the energy of mean flows. Here a critical examination of these favorably configured perturbations is given, making use of the simple constant free shear barotropic model which allows construction of exact two-dimensional isolated wave packet solutions.

1. Introduction

Mobile synoptic-scale disturbances in the atmosphere are commonly understood to arise from instabilities of the baroclinic zonal flow (Charney, 1947; Eady, 1949). Such instability theories are widely invoked in fluid dynamics to account for the observed breakdown of laminar flow into wavelike and ultimately turbulent motion (Drazin and Reid, 1981). It is argued that if the nonlinear equations linearized about a stationary solution support normal modes which grow exponentially in time, the mode with the largest growth rate must ultimately dominate the response to small but presumably nonzero perturbations which contaminate at some level even the most carefully prepared approximation to laminar flow. The asymptotic dominance of the fastest-growing mode implies that the precise configuration of the initial condition is irrelevant because a more slowly growing mode is presumed unlikely to be overwhelmingly favored in the perturbation. This together with the result that the contribution of the complement continuous spectrum in simple model problems can be shown to decay algebraically in the limit of long time (Thomson, 1887; Case, 1960; Yamagata, 1976) has reduced the study of development in shear flow to the identification of the fastest-growing normal modes.

Recent work in the theory of cyclogenesis (Farrell, 1982, 1984, 1985) has focused attention on initial conditions which intensify whether or not the flow supports instabilities. In the event that instabilities are present, their growth rates are often found to be small compared to that of a suitably chosen energy releasing perturbation. Whereas exponential instability theory places little emphasis on the initial disturbance, in the initial

value problem the disturbance configuration is crucial to the development. Directing attention to those configurations which subsequently develop is natural from the forecasters point of view and forecast rules often correspond to identifying them (Palmen and Newton, 1969, chap. 11).

The investigation of initial growth in the Couette problem made by Orr (1907) showed that even in this example which supports no normal modes, stable or unstable, it is possible to identify perturbations which increase in disturbance energy by an arbitrary amount. For the example of unbounded constant shear, this increase was shown to be unchanged by the inclusion of the β effect (Yamagata, 1976). The implication of these results is strengthened for examples which support linear normal modes by the observation that energy released during the initial growth phase becomes trapped in the modes (Farrell, 1982, 1984; Held, 1985); this provides a means of exciting neutral modes and obviates the asymptotic decay which led to the conclusion, based on the constant-free shear and Couette problems, that only exponential normal modes can produce a nonzero asymptotic contribution to disturbances amplitude. In this context it is interesting to observe that even in the Couette problem of Orr (1907) the nonlinear modification of the shear by the perturbation can lead to persistent normal modes (Hou and Farrell, 1986).

There are exact solutions of initial value problems for constant-free shear and the Couette problem (Orr, 1907) as well as for the Eady problem (Farrell, 1984). In these examples the elementary disturbance assumed is a plane wave and the amplification is approximately proportional to α^2 , the ratio of the cross-stream to longstream wavenumber and positive if the wave phase

lines are oriented against the shear. (The Eady problem stabilized by confining the wave to a sufficiently narrow channel behaves analogously to the Couette problem during the initial growth phase.) Because the amplification is nearly proportional to α^2 , it is clear how to choose a perturbation which undergoes any prescribed growth: a plane wave leaning against the shear with sufficiently high cross-stream wavenumber.

It may be argued that a single wave is an unlikely initial condition and that exponential instability would dominate a reasonable development from random initial conditions. Here there are a number of issues. First, except in the case of the constant-free shear, where there are no modes, the development of the (stable or unstable) normal mode is inextricably linked in its energetics to continuous spectrum modes which complete the modal projection of the perturbation. This is reflected in the fact that these modes are not orthogonal (Farrell, 1984). Additionally, the growth rate of unstable waves in the presence of realistic dissipation or long stream inhomogeneity is so small (Farrell, 1985; Pierrehumbert, 1984; Card and Barcion, 1982) that if observed developments are to be ascribed to shear energetics they are quite likely the product of transient growth. However, granting that a plane wave can develop arbitrarily it remains to show that more realistic perturbations share this growth, in particular, a disturbance localized in space. Dispersive effects could, for instance, spread out a local disturbance more rapidly than it grew. In order to examine this possibility a plane wave disturbance in the Eady model was confined to a single wavelength in the zonal direction and it was found that the development was essentially unaffected (Farrell, 1985). In the same report, a plane wave was truncated to a single scale height in the Charney problem to examine the effect of cross-stream confinement of the perturbations and the growth was again found to be substantially unaffected. While these examples suggest the phenomena of robust transient development is not confined to plane wave initial conditions, a more thorough look at the characteristics of developing disturbances is appropriate.

2. The model problem

The purpose of understanding transient development is best served by choosing the simplest example which supports the phenomena: constant-free shear. Neglect of the β effect seems to be justified by observing that the amplitude of individual plane waves is not a function of β (Yamagata, 1976). However, the phase and therefore the composite structure as a function of time of an arbitrary initial perturbation is affected and it is simple to include β where appropriate. More restrictive is the neglect of boundaries and the normal modes which they support as these have been shown to qualitatively change the asymptotic as well as the transient response (Farrell, 1984). However, internal

perturbation growth is of interest in its own right, and the presence of boundaries serves mostly to modify the short-term transient development in which we are interested.

The argument to follow is made for barotropic flow; however, the results apply as well to the baroclinic quasi-geostrophic problem with a suitable identification of variables (Farrell, 1982) so that the original purpose of gaining insight into the process of cyclogenesis is also served, although the lack of boundary-supported modes is perhaps a more severe approximation in the baroclinic problem.

The linearized barotropic vorticity equation for a zonal flow is

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) \nabla^2 \psi + (\beta - U_{yy}) \psi_x = 0 \quad (1)$$

where U is the zonal mean velocity and ψ the streamfunction of the perturbation. The equation is restricted to a constant shear, $U = Sy$ so that (1) becomes:

$$\left(\frac{\partial}{\partial t} + Sy \frac{\partial}{\partial x}\right) \nabla^2 \psi + \beta \psi_x = 0. \quad (2)$$

It will aid understanding of solutions to (1) to have the perturbation energy equation at second order (Pedlosky, 1979):

$$\frac{\partial}{\partial t} \iint \frac{u^2 + v^2}{2} dx dy = \iint (uv)_y U dx dy, \quad (3)$$

in which the density is set to unity.

Here the Reynolds stress divergence $-(uv)_y$ is interpreted as a force on the mean flow. This product integrated over the region occupied by the wave is the loss of mean energy which must balance the gain in perturbation energy. It follows from an integration by parts of (3) that a perturbation which is gaining energy must have the quantity

$$-(uv) \frac{\partial U}{\partial y} = \psi_y \psi_x \frac{\partial U}{\partial y}$$

greater than zero on average which implies:

$$\frac{\psi_x (\psi_y)^2}{\psi_y} \frac{\partial U}{\partial y} > 0$$

$$-\left(\frac{\partial y}{\partial x}\right) (\psi_y)^2 \frac{\partial U}{\partial y} > 0.$$

This is the familiar requirement that developing perturbations on average have lines of constant phase with slope opposite to that of the shear, i.e., that "lean against" the shear.

Practically, this characteristic orientation of phase surface is a useful guide to identifying developing disturbances. In particular, the forecaster is alert to the westward tilt implied for growing modal or nonmodal

baroclinic waves. While it is easy to find symmetric perturbations which develop tilts under the influence of a shear (Farrell, 1982; Shepherd, 1985), these do not appear to be as common in observations (Palmén and Newton, 1969).

3. Plane wave solution

It may now be verified that (2) has the solution for the plane wave initial perturbation $\psi(t=0) = Ae^{i(k_0x+l_0y)}$ (Yamagata, 1976):

$$\psi = A \frac{k_0^2 + l_0^2}{k_0^2 + (l_0 - Sk_0t)^2} e^{i(k_0x + (l_0 - Sk_0t)y + \phi)} \quad (4a)$$

$$\phi = \frac{\beta}{Sk_0} \left\{ \tan^{-1} \left(\frac{l_0}{k_0} \right) - \tan^{-1} \left(\frac{l_0}{k_0} - St \right) \right\}. \quad (4b)$$

The amplitude of the streamfunction is independent of β but the phase and therefore the phase speed and group velocity are functions of β (Yamagata, 1976; Tung, 1983).

The perturbation velocities are

$$u = -\psi_y = -i(l_0 - Sk_0t)\psi$$

$$v = \psi_x = ik_0\psi.$$

Physical quantities are understood to correspond to the real parts of the above.

The phase averaged perturbation kinetic energy density is

$$\begin{aligned} E(t) &= \frac{A^2}{4} \frac{(k_0^2 + l_0^2)^2}{k_0^2 + (l_0 - Sk_0t)^2} \\ &= E(0) \frac{k_0^2 + l_0^2}{k_0^2 + (l_0 - Sk_0t)^2}. \end{aligned} \quad (5)$$

The ratio of kinetic energy at a later time to its initial value reaches a maximum for $(S, k_0, l_0) > 0$ of

$$\frac{E(l_0/Sk_0)}{E(0)} = 1 + \left(\frac{l_0}{k_0} \right)^2.$$

Asymptotically as $t \rightarrow \infty$ the energy of an individual plane wave component decays as t^{-2} but it is clear that the perturbation growth can be made arbitrarily large, being the square ratio of the meridional to zonal initial wavenumbers.

There are some issues of interest concerning this energy growth. The first is in regard to the asymptotic decay. The inverse shear time scale in the midlatitude atmosphere is typically ~ 10 h (Farrell, 1984). This is also the typical time scale for cyclogenesis (Palmén and Newton, 1969) so that the $t \rightarrow \infty$ asymptotic is often physically irrelevant and it is time scales for which the behavior is dominated by the initial growth that are relevant. In addition, it is important to notice that the asymptotic decay in this and the Couette problem is exceptional and rather unfortunate as it has led to the

conclusion that the interaction of the modal and non-modal waves can be neglected. It is only because there are no normal modes in these problems that there is asymptotic decay. In examples with neutral, near neutral, or nonlinear modes perturbations gain energy from the mean and deposit it in the modes where it persists (Farrell, 1985; Hou and Farrell, 1986; Held, 1985). Moreover, in section 5 a broad spectrum initial condition is shown not to exhibit this asymptotic decay. In summary, while the transient growth is robust, the t^{-2} asymptotic decay is not.

A further question related to energetics concerns the growth rate compared with normal mode growth rates. Examples have shown that over the time and space scales of synoptic developments, transient growth rates far exceed exponential mode growth rates (Farrell, 1982, 1985). As there are no modes in the free shear problem it is instructive to compare the growth in this case with integral bounds.

From an integration by parts of (3) energy growth is limited by

$$\frac{\partial}{\partial t} \langle E \rangle \leq S \langle (|u||v|)_{\max} \rangle \leq S \langle E \rangle$$

so that

$$\frac{\partial}{\partial t} \ln \langle E \rangle \leq S, \quad (6)$$

where brackets signify integration over the domain of the wave.

For comparison, the maximum growth rate in the Eady problems is $0.620S$ and a wave of aspect ratio one grows at a maximum of $0.438S$; furthermore, this maximum occurs only at a single value of wavenumber with a rapid falloff in growth at other wavenumbers. Similar results for the Charney problem are $0.572S$ and $0.405S$. Typical of barotropic problems is the maximum growth rate of the split-line velocity profile, $0.402S$ (Gill, 1982, p. 567).

By comparison, the plane wave growth rate from (5) is

$$\frac{\partial \ln E}{\partial t} = 2S \frac{\frac{l_0}{k_0} - St}{1 + \left(\frac{l_0}{k_0} - St \right)^2}.$$

The growth rate is a maximum at $t = (l_0/k_0 - 1)/S$ one shear time unit prior to the energy maximum at $t = (l_0/k_0)/S$ and at a time when the wave is tilted $\theta = \tan^{-1}(l_0 - k_0St)/k_0 = 45^\circ$. Remarkably, the maximum equals the limit set by the energy bound (6) which is usually regarded as a loose upper bound. Furthermore, the limiting growth rate is obtained at every wavenumber, whereas an exponential mode obtains a lesser rate only at a specific wavenumber. Various developing initial conditions in other model problems with boundaries give smaller maxima but generally far in excess of normal mode values (Farrell, 1982, 1985).

The average growth rate over the interval of development may be estimated using (5):

$$\frac{\partial \ln E}{\partial t} \approx \frac{\Delta \ln E}{\Delta t} = S \frac{\ln(1 + (l_0/k_0)^2)}{l_0/k_0}$$

Results are collected for representative aspect ratios $\alpha = l_0/k_0$ in Table 1. Average growth rates are seen to exceed maximum exponential mode values obtained in other problems over a wide range of α .

As is the case with other model problems for which closed form solutions are possible, the plane wave in free shear has peculiarities associated with boundary terms which obscure its energetics. In particular (4):

$$\overline{uw} = -\frac{A^2 k_0^2}{2} (1 + \alpha^2)^2 \frac{(\alpha - St)}{[1 + (\alpha - St)^2]^2} \quad (7)$$

with which the derivative of energy (5) is

$$\frac{\partial E}{\partial t} = -\overline{uw}S.$$

Although this appears to be consistent energetically when integrated over the plane (\overline{uw})_y vanishes by (7) so that the mean flow relation

$$\frac{\partial U}{\partial t} = -(\overline{uw})_y$$

implies no interaction in the interior of the plane wave solution. Resolution of this quandary is immediate on noticing that the integration by parts of (3) involves boundary terms not accounted for in the unbounded plane wave solution. The problem is rendered energetically consistent by including the mean flow interaction which occurs at the edges of the wave packet. Physically, a plane wave in constant shear transports momentum without divergence to its boundaries where the mean flow interaction occurs. Examples in section 4 will make this result explicit; the argument is stressed because familiarity with the plane wave solution may obscure the essential identity of the shear energetics of unstable waves and developing perturbations in initial value problems. In particular, it can lead to the conclusion that the plane wave is a nonlinear solution, a result which is valid only when the mean flow interactions, which must occur at the boundaries of a physical wave packet, are ignored. Examples of wave-mean

TABLE 1. Maximum normalized energy increase and average growth rate in units of the shear, S , for plane waves and wave packets (bracket).

α	E/E_0	$(\Delta \ln E)/\alpha$
1.0	2.0 (1.75)	0.70 (0.56)
2.0	5.0 (3.3)	0.80 (0.60)
4.0	17.0 (7.4)	0.71 (0.50)

flow interaction at small and large amplitude may be found in Hou and Farrell (1985).

4. More general perturbations

Fourier synthesis provides the means of extending the plane wave solution to include arbitrary physically realizable initial disturbances. The spectrum of waves in the general linear problem may include a set with phase lines oriented opposite the shear for which $k, l \geq 0$ and $\partial E/\partial t \geq 0$ together with a set oriented with the shear for which $l < 0$ and $\partial E/\partial t < 0$. Therefore, at a given time there may be both growing and decaying components and the overall energetics at that time depend on which is dominant. Moreover, the balance of power changes with time and the shear may produce a growing disturbance for which $\langle \overline{uw} \rangle < 0$ from a decaying one for which $\langle \overline{uw} \rangle > 0$. The simplest example of this is the checkerboard perturbation (Boyd, 1983):

$$\begin{aligned} \psi &= A \cos(k_0 x) \cos(l_0 y) \\ &= \frac{A}{2} [\cos(k_0 x + l_0 y) + \cos(k_0 x - l_0 y)] \end{aligned}$$

made up of the sum of a growing wave and a decaying wave for which initially $\overline{uw} = 0$. Subsequently $\overline{uw} < 0$ as the growing component temporarily dominates the transient and the perturbation acquires an average tilt against the shear so that $\partial E/\partial t > 0$. Such examples are of theoretical interest but should not be allowed to obscure practical use of the Reynolds stress diagnostic: developing perturbation, modal or nonmodal, lean against the shear. Both in observation and model studies a robustly developing wave has quite generally to the eye, an unmistakable tilt so that the forecast need seldom turn on subtleties.

The energy of each component wave behaves asymptotically as t^{-2} by Eq. (5). This has led to the conclusion that almost all composite perturbations in shear flow decay and it is highly exceptional to find one with robust growth. However, it is easy to demonstrate making use of the time reversal symmetry of the inviscid equation that there is a one-to-one correspondence between decaying perturbations and those which grow arbitrarily large. The Fourier integral which represents the decaying disturbance as $t \rightarrow \infty$ is over components each of form (4). At an advanced time $\tau > 0$ when the original disturbance, now decaying as t^{-2} , has reached as small an amplitude as is desired, the meridional wavenumber of a component is $-\lambda_0 = l_0 - k_0 S \tau$. Form a new growing perturbation by reflecting the phase lines from the decaying to the growing orientation $-\lambda_0 \rightarrow +\lambda_0$. The perturbation now leans against the shear with components:

$$\psi = A \frac{k_0^2 + l_0^2}{k_0^2 + (\lambda_0 - S k_0 t)^2} e^{i(k_0 x + (\lambda_0 - S k_0 t) y)}$$

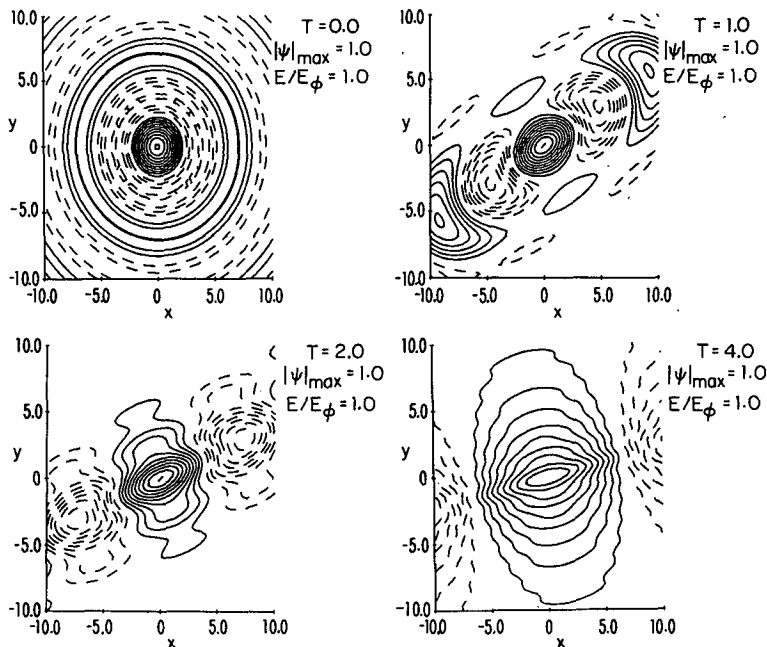


FIG. 1. Development of the ring δ -function perturbation which at the initial time corresponds to $J_0(r)$. Maximum amplitude and normalized energy are indicated for (a) $t = 0.0$, (b) $t = 1.0$, (c) $t = 2.0$ and (d) $t = 4.0$.

Each component now retraces its decay curve in the opposite direction in time growing by the same arbitrary amount by which it decayed. It is clear that decaying perturbations are no more difficult to find than growing, there being a one-to-one correspondence. Additionally, a full spectrum perturbation which grows by an arbitrary factor can be produced by reflecting any initial perturbation which has decayed by that same factor.

5. Wave packets

Examples of perturbation evolution illustrating the process of transient development in free shear follow.

The ring δ -function which consists of an equal excitation of all orientations of a single wavenumber has been referred to as a general perturbation and the lack of energy growth in this example advanced as an argument against the transient development phenomena (Shepherd, 1985). Writing the 2 - d transform

$$\psi(x, y) = \frac{1}{2\pi} \int \int_{-\infty}^{\infty} \hat{\psi}(k, l) e^{i(kx+ly)} dk dl \quad (8)$$

in polar coordinates and taking advantage of the circular symmetry gives

$$\psi(r) = \int_0^{\infty} \varphi \hat{\psi}(\varphi) J_0(\varphi r) d\varphi$$

$$x = r \cos\theta$$

$$y = r \sin\theta$$

$$k = \varphi \cos\phi$$

$$l = \varphi \cos\phi.$$

Taking $\hat{\psi}(\varphi) = \delta(\varphi - \varphi_0)$ results in the transform:

$$\psi(r) = \varphi_0 J_0(\varphi_0 r).$$

Although this perturbation appears plausibly when viewed in Fourier space it corresponds in physical space to the Bessel function of zero order, an infinite "bull's eye" of counterrotating bands, Fig. 1a.¹ In addition, the exact circular symmetry suggests no systematic tilt and correspondingly no growth. This is confirmed by the energy integral (Shepherd, 1985) which reveals no overall increase in perturbation energy. This example underscores the utility of examining the perturbation field in physical space. The theory of initial value problems gives the forecaster specific aid: he is to look for energy releasing perturbations which are identified by phase tilt opposite the shear. Disturbances such as Fig. 1a clearly are not candidates for development. Cyclogenesis in the atmosphere is an episodic occurrence despite the observed energetic perturbation field pre-

¹ Shepherd (personal communication) has pointed out that the imposition of a random phase in the Fourier components of (8) gives a random wave field (with vanishing amplitude as normalized above) which fills space rather than an isolated coherent disturbance. As previous work addressed localized coherent initial conditions, it is consistent with the cyclogenesis problem to use the inversion in Fig. 1. Energetics is not affected by this choice nor is the behavior of the spectrum as a function of time.

cisely because most disturbances are not strongly energy releasing.

While the ring δ -function does not gain energy from the mean flow, it is a good model of phenomena related to the maintenance of the perturbation spectrum in shear. A thorough investigation of this problem would be aside from the main topic of this work and must involve the accumulation of energy in modes associated with boundaries and jets; nevertheless, a comment on the process is appropriate as an aid in interpreting the foregoing example.

Explanation of observed spectra in the atmosphere and in fluid flow in general is provided by turbulence theory which is well developed and will not be reviewed here (Charney, 1971). Less effort has been given to perturbations which occur in flows dominated by a mean shear, although an early use of transformed coordinates to solve the initial value problem (Phillips, 1966) was motivated by an attempt to explain the sheared internal wave spectra. The physical mechanism involved is clear from an examination of the amplitude relation (4): in a field of sheared waves the maximum amplitude of each component occurs coincident with its minimum of wavenumber. In particular, an isotropic distribution of components produces an accumulation of energy at small wavenumbers which corresponds to a red spectrum.

An approximate solution for the ring δ -function which captures many of the features of this behavior at large St is obtained in appendix A:

$$\psi(x, y, t) \approx \varphi_0 \cos\left(\frac{\varphi_0}{(St)^2}(Stx + y)\right) e^{-[\varphi_0/(St)^2]|x-sty|}. \quad (\text{A.1})$$

As St becomes large the wavelength of the dominant harmonic increases linearly with time while the wave assumes a zonal orientation. An exponential decay characterizes the cross-stream amplitude and a local shear line forms across the "critical level" $x - sty = 0$. Figure 1 shows some details of these phenomena for the example of the ring δ -function with $\varphi_0 = 1.0$; in these figures the maximum amplitude of the streamfunction is indicated at each time in order that the resolution not be compromised by a constant contour interval. The algorithms used to produce this and subsequent figures is a direct Riemann sum of (8) using N harmonics evenly distributed on the boundary in this case or over the interior of the two-dimensional figures to follow. The complex amplitude of each wave as a function of time is obtained from Eq. (4). A total of $N = 36$ was sufficient for these short time solutions.

Based on this example a source of randomly oriented fluctuations in a shear flow can be expected to produce an accumulation of energy at low wavenumber corresponding to a red spectrum. Such randomly oriented waves will not, however, possess the correlation of ve-

locities which give rise to large average Reynolds stress and systematic release of mean flow energy.

In order to examine energy releasing perturbations localized in space, a more realistic modulation than the ring δ -function is needed. A Gaussian of the form

$$\psi(k, l) = \frac{R^2}{2} e^{-(R^2/4)(k^2+l^2)}$$

has transform

$$\psi(x, y) = e^{-((x^2+y^2)/R^2)}.$$

This perturbation is also circular symmetric with no overall growth in time. The concentration of energy at small wavenumber discussed here is evident in Fig. 2 which shows results for $R = 2.0$.

The prototype of a developing perturbation is the plane wave:

$$\psi(k, l) = \cos(kx + ly).$$

with $k = 1.0$, $l = 2.0$, the energy from (5) increases by a factor of 5.0.

Putting these examples together a developing two-dimensional localized perturbation can be formed as the product:

$$\psi(x, y, t = 0) = e^{-((x^2+y^2)/4)} \cos(x + 2y).$$

Solutions for wave packets in the WKB approximation (Yamagata, 1976; Tung, 1983) predict central wave development following (4) and energy variation according to (5); however, the narrowly isolated wavepackets of interest in cyclogenesis do not satisfy WKB slow variation requirements and the removal of this restriction here provides insight regarding the performance of the WKB approximation.

Subsequent growth (Fig. 3), although reduced, is similar to that found in the plane wave. However, the $t \rightarrow \infty$ decay is intercepted by the tendency toward a red spectrum discussed previously in regard to the modulation function. Average growth rates for this and wave packets with $\alpha = 1.0$ and 4.0 (for which figures are not shown) are indicated in the parentheses of Table 1.

Reynolds stress divergence in the wave packet is correlated according to equation (3). Figure 4 shows $(uv)_y$ for the example of Fig. 3; unlike the plane wave for which $(uv)_y = 0$, and all interactions takes place at infinity, the wave packet tendency is to modify the shear so as to release energy during growth and return it during decay. Notice that the effect of the wave is to bend the mean flow into an "s" profile during the growth phase (4a-4c) and into a reverse "s" during the decay phase (4d); see Hou and Farrell (1986) for examples of this nonlinear interaction.

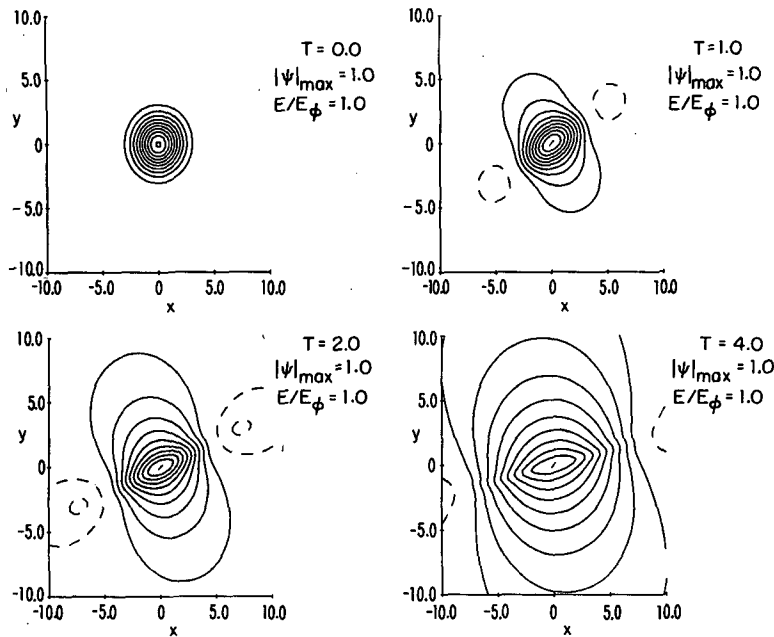


FIG. 2. As in Fig. 1 but for the Gaussian modulation function. Maximum amplitude and normalized energy are indicated for (a) $t = 0.0$, (b) $t = 1.0$, (c) $t = 2.0$ and (d) $t = 4.0$.

6. Discussion

The linearized equation which describes the behavior of two-dimensional fluid flow and the similar linear quasi-geostrophic equation can support normal modes

which grow exponentially in time if favorable flow configurations and boundary conditions are present. As these modes dominate the $t \rightarrow \infty$ asymptotic and the contribution of nonmodal waves has been shown to decay in this limit in two simple examples, constant-

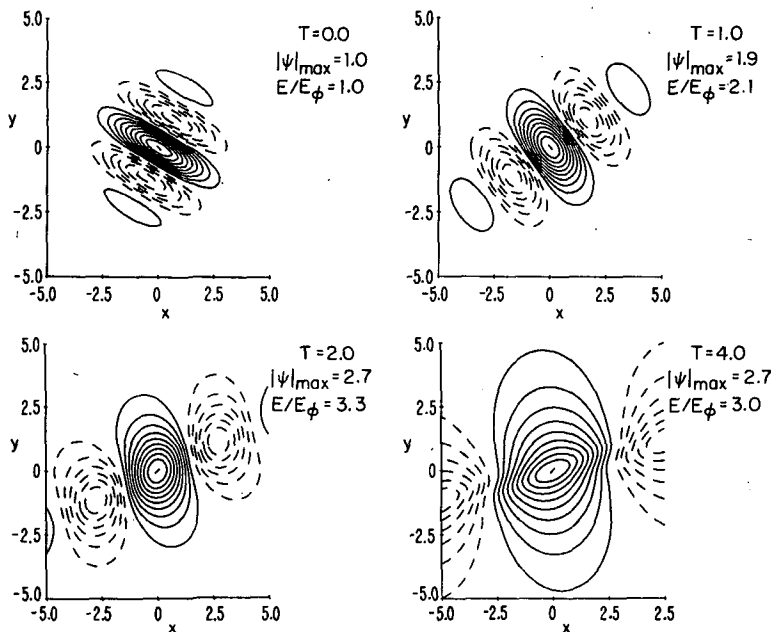


FIG. 3. As in Fig. 1 but for the composite wave packet. Maximum amplitude and normalized energy are indicated for (a) $t = 0.0$, (b) $t = 1.0$, (c) $t = 2.0$ and (d) $t = 4.0$.

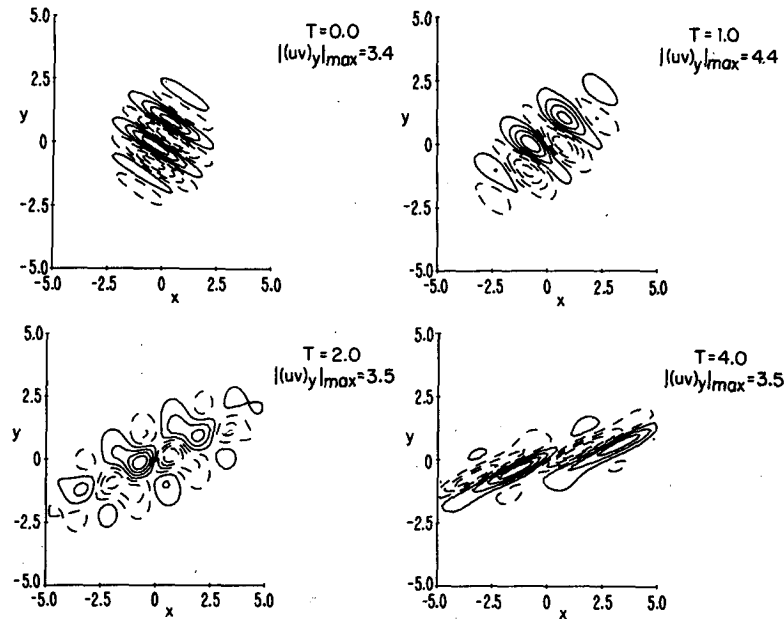


FIG. 4. Reynolds stress divergence $(uv)_y$, for the composite wave packet in Fig. 3. Maximum amplitude and normalized energy are indicated for (a) $t = 0.0$, (b) $t = 1.0$, (c) $t = 2.0$ and (d) $t = 4.0$.

free shear and the Couette problem, development of perturbations in shear has been widely ascribed to instabilities. However, it has been known since Orr (1907) that instabilities are not necessary to produce arbitrary development and that it is possible to obtain a non-modal perturbation which grows to any desired extent, at least in the linear approximation. Development in the atmosphere takes place on scales of a few inverse shear times which is appropriate for initial value problems rather than for $t \rightarrow \infty$ asymptotics and there are a number of examples of IVP cyclogenesis (Farrell, 1982, 1984, 1985). Unlike the case of normal modes, for which the eigenmode associated with the greatest exponential growth provides the structure of the disturbance, the initial value problem provides less information on the form of the growing wave. This investigation returned to the simple example of constant free shear to study these perturbations; among the results are:

(i) Developing perturbations have upshear tilts which are clear, when viewed in physical space, and which provide the basis for forecasting growth.

(ii) Every disturbance which decays can be associated with a companion which grows arbitrarily and has the same spectrum by changing the sign of the cross stream wavenumber.

(iii) Nonmodal growth rates can exceed typical normal mode rates on average over the development and equal episodically the theoretical maximum energy integral bound.

(iv) Although the plane wave solution is not energetically consistent unless some account of boundary interaction is made, wave packets have energetics similar to unstable waves with Reynolds stress divergences acting to reduce mean flow energy during perturbation development.

(v) An isotropic excitation of waves has no average Reynolds stress correlation so that total perturbation energy is constant. However, the wave spectrum is not constant as energy is continually concentrated at small wavenumber producing a red spectrum. Thus, while the WKB approximation (Tung, 1983) gives qualitatively correct results for the growth phase of narrowly isolated wave packets, their broad spectrum causes the predicted $t \rightarrow \infty$ decay to be intercepted by an accumulation of energy at ever smaller wavenumbers which may greatly reduce the rate of perturbation decay even in the absence, as here, of modal solutions.

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APPENDIX A

Approximate Solution for the Ring δ -function

At any time the streamfunction can be expressed as the Fourier integral over the component waves:

$$\psi(x, y, t) = \frac{\varphi_0^2}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i(kx+ly)}}{k^2 + l^2} \frac{\delta(k - k_c)\delta(l - l_c)}{|dS_c/dS|} dk dl$$

$$k_0^2 + l_0^2 = \varphi_0^2$$

$$k_c = k_0$$

$$l_c = l_0 - Stk_0.$$

The factor $|dS_c/dS|$ modifies the strength of the δ according to the deformation of the ring:

$$(dS_c)^2 = (dk)^2 + (dl)^2 \left(1 - St \frac{dk_0}{dl_0}\right)$$

$$(dS)^2 = (dk)^2 + (dl)^2$$

$$\frac{dS_c}{dS} = \left[\frac{1 + (dl_0/dk_0 - St)^2}{1 + (dl_0/dk_0)^2} \right]^{1/2}.$$

As $St \rightarrow \infty$, the dominant contribution occurs along two line segments which can be parameterized by ϵ (neglecting curvature effects) as $\pm k_c = l_0/St + \epsilon \approx \varphi_0/St + \epsilon$, $\pm l_c = l_0 - Stk_0 = -St\epsilon$. The area integral is converted to a line integral in this limit by:

$$dS_c = [(dk_c)^2 + (dl_c)^2]^{1/2} = d\epsilon [1 + (St)^2]^{1/2} \approx St d\epsilon$$

and the deformation factor is in the same limit $|dS_c/dS| \approx St$ so that the integral becomes:

$$\psi(x, y, t) = \frac{\varphi_0^2}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(kx+ly)}}{k^2 + l^2} \{ \delta[k - (\varphi_0/St + \epsilon)]\delta(l + St\epsilon) + \delta[k + (\varphi_0/St + \epsilon)]\delta(l - St\epsilon) \} d\epsilon$$

$$\approx \frac{1}{\pi} \left(\frac{\varphi_0}{St}\right)^2 \text{Re} \int_{-\infty}^{\infty} \frac{e^{i[(\varphi_0/St)x + \epsilon(x - Sty)]}}{[\varphi_0/(St)^2]^2 + \epsilon^2} d\epsilon.$$

The poles at $\epsilon \approx -\varphi_0/(St)^3 \pm i\varphi_0/(St)^2$ are used to evaluate the integral:

$$\psi(x, y, t) \approx \frac{1}{\pi} \left(\frac{\varphi_0}{St}\right)^2 \times \text{Re} \left\{ \frac{2\pi i}{2i\varphi_0/(St)^2} e^{i[(\varphi_0/St)x + (\varphi_0/(St)^2)y]} e^{-(\varphi_0/(St)^2)|x - Sty|} \right\}$$

$$\psi(x, y, t) \approx \varphi_0 \cos\left(\frac{\varphi_0}{(St)^2}(Stx + y)\right) e^{-(\varphi_0/(St)^2)|x - Sty|}. \tag{A1}$$

This approximation is adequate in the near field shown in Fig. 1. However, delta function perturbations in Fourier space correspond to unbounded energy in unbounded space domains and although the point delta is characterized by a constant energy density leading to a natural normalization, the distribution of energy in the far field is controlled by the curvature in the case of the line delta and there is an unbounded energy lying outside of any bonded domain. Curvature of the line segments in the above representation could be included to render it asymptotic in the far field but this greatly complicates the analysis with little insight gain.

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