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OPTIMAL PERTURBATION OF UNCERTAIN SYSTEMS

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In studies of perturbation dynamics in physical systems, certain specification of the governing perturbation dynamical system is generally lacking, either because the perturbation system is imperfectly known or because its specification is intrinsically uncertain, while a statistical characterization of the perturbation dynamical system is often available. In this report exact and asymptotically valid equations are derived for the ensemble mean and moment dynamics of uncertain systems. These results are used to extend the concept of optimal deterministic perturbation of certain systems to uncertain systems. Remarkably, the optimal perturbation problem has a simple solution: In uncertain systems there is a sure initial condition producing the greatest expected second moment perturbation growth.

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1. Introduction

Linear stability theory has been extensively studied because of its role in advancing understanding of a wide range of physical phenomena related to the structure and growth of perturbations to dynamical systems. Historically, linear stability theory was developed using the method of modes (Rayleigh [13]). However, the method of modes is incomplete for understanding perturbation growth even for autonomous systems because the non-normality of the linear operator in physical problems often produces transient development of a subset of perturbations that dominates the physically relevant growth processes (Farrell [4]). Recognition of the role of non-normality in linear stability led to the development of generalized stability theory (Farrell and Ioannou [6]). Compared to the methods of modes, the methods of generalized stability theory, which are based on the non-normality of the linear operator, allow a far wider class of stability problems to be addressed including perturbation growth associated with aperiodic time-dependent certain operators to which the method of modes does not apply (Farrell and Ioannou [7]).

The problems to which generalized stability theory has been applied heretofore involve growth of perturbations in a system with no time dependence or a system with known time dependence; perturbations to these systems may be sure or stochastically distributed and may be imposed at the initial time, or distributed continuously in time, but the operator to which the perturbation is applied is considered to be certain and the problem is that of the stability of a certain operator. However, it may happen that we either do not have complete knowledge of the system that is being perturbed or that exact specification may be inappropriate to the physical system in which case the problem is to determine the stability of an uncertain operator. In generalized stability theory the perturbation producing greatest growth plays a central role in quantifying the stability of the system. In this paper we obtain exact dynamical equations for the perturbation ensemble mean and covariance and use these results to obtain the optimal perturbation to uncertain systems.

2. Exact Equations for the Ensemble Mean

Consider the uncertain linear system for the vector state ψ:

$$\frac{d\psi}{dt} = \mathbf{A}\psi + \varepsilon\eta(t)\mathbf{B}\psi\,, \tag{2.1}$$

where $\mathbf{A}$ is the ensemble mean matrix, $\eta(t)$ is a stochastic process with zero mean, $\mathbf{B}$ is the matrix of the fluctuation structure and ε is an amplitude parameter. Equations for the evolution of the ensemble mean field, $\langle\psi(t)\rangle$, and for the ensemble mean covariance, $\langle\psi_i(t)\psi_j^*(t)\rangle$, can be readily obtained if η is a white noise process (Arnold [1]). Although it is a great advantage for analysis to assume that η is a white noise process, this assumption is often hard to justify in physical contexts because the uncertainties in physical operators are often red and because white noise processes producing nonvanishing variance in (2.1) have unbounded fluctuations that in a physical context would imply, for example, infinite wind speeds or unbounded negative damping rates.

Approximate dynamical equations for the ensemble mean field evolving under uncertain dynamics and not restricted to white noise stochastic processes have been obtained by Bourret [2], Keller [8], Papanicolaou and Keller [11] and Van Kampen [14]. These approximations are accurate for small Kubo number $K = \varepsilon t_c$, where t_c is the autocorrelation time of the stochastic process $\eta(t)$. In addition, exact evolution equations for the ensemble mean can be obtained when $\eta(t)$ is a telegraph process; results which are discussed in Brissaud & Frisch [3]. Finally, making use of the properties of cumulant expansions it is possible to obtain an exact expression for the evolution of the ensemble mean covariance for the case in which $\eta(t)$ is a Gaussian process as follows:

Theorem 2.1. *If $\eta(t)$ is a Gaussian stochastic process with zero mean, unit variance, and autocorrelation time $t_c = 1/\nu$ so that $\langle\eta(t_1)\eta(t_2)\rangle = \exp(-\nu|t_1 - t_2|)$ then the ensemble mean of the stochastic equation* (2.1) *obeys the deterministic equation*:

$$\frac{d\langle\psi\rangle}{dt} = (\mathbf{A} + \varepsilon^2\mathbf{BD}(t))\langle\psi\rangle\,, \tag{2.2}$$

where

$$\mathbf{D}(t) = \int_0^t e^{\mathbf{A}s}\mathbf{B}e^{-\mathbf{A}s}e^{-\nu s}ds\,. \tag{2.3}$$

Proof. Consider the interaction perturbation $\phi(t)$ defined by

$$\psi(t) \equiv e^{\mathbf{A}t}\phi(t)\,, \tag{2.4}$$

in which Eq. (2.1) becomes:

$$\frac{d\phi}{dt} = \varepsilon\eta(t)e^{-\mathbf{A}t}\mathbf{B}e^{\mathbf{A}t}\phi = \varepsilon\eta(t)\mathbf{H}(t)\phi\,, \tag{2.5}$$

where

$$\mathbf{H}(t) \equiv e^{-\mathbf{A}t}\mathbf{B}e^{\mathbf{A}t}\,. \tag{2.6}$$

For a sure initial perturbation we have $\phi(0) = \psi(0)$ and the interaction perturbation at time t is:

$$\phi(t) = \mathbf{G}(t)\psi(0)\,, \tag{2.7}$$

with the fundamental matrix $\mathbf{G}(t)$ given by

$$\mathbf{G}(t) = \mathbf{I} + \varepsilon\int_0^t \eta(t_1)\mathbf{H}(t_1)dt_1 + \varepsilon^2\int_0^t dt_1\int_0^{t_1} dt_2\eta(t_1)\mathbf{H}(t_1)\eta(t_2)\mathbf{H}(t_2) + \cdots\,. \tag{2.8}$$

This expression for the fundamental matrix defines the time-ordered exponential:

$$\mathbf{G}(t) = \exp_{\mathrm{o}}\left(\varepsilon\int_0^t \eta(s)\mathbf{H}(s)ds\right). \tag{2.9}$$

The subscript "o" denotes time ordering and the term exponential is used because Eq. (2.8) is obtained if the argument of Eq. (2.9) is expanded as an exponential with the convention that the terms are grouped in ascending time order. Note that time ordered exponentials of matrices satisfy the familiar properties of exponentials of scalars even for matrices that do not commute; for example, we can write $\exp_{\mathrm{o}}(\mathbf{H}(t_1) + \mathbf{H}(t_2)) = \exp_{\mathrm{o}}(\mathbf{H}(t_1))\exp_{\mathrm{o}}(\mathbf{H}(t_2))$ even if the matrices $\mathbf{H}(t_1)$ and $\mathbf{H}(t_2)$ do not commute if time ordering is enforced meaning that in all products $\mathbf{H}(t_1)$ is placed to the left of $\mathbf{H}(t_2)$.

Kubo [9] has shown that the average of the time ordered exponential in Eq. (2.9) can be expanded in the cumulants of $\eta(t)$ in the same way that the exponents of ordinary stochastic processes are expanded in cumulants of their arguments:

$$\langle\mathbf{G}(t)\rangle = \exp_{\mathrm{o}}(\mathbf{F}(t))\,, \tag{2.10}$$

where

$$\mathbf{F}(t) = \frac{\varepsilon^2}{2!}\int_0^t\int_0^t dt_1 dt_2 \ll \eta(t_1)\eta(t_2) \gg \mathbf{H}(t_1)\mathbf{H}(t_2) + \cdots$$
$$+ \frac{\varepsilon^{2n}}{n!}\int_0^t \cdots \int_0^t dt_1 \cdots dt_n \ll \eta(t_1)\cdots\eta(t_n) \gg \mathbf{H}(t_1)\cdots\mathbf{H}(t_n) + \cdots\,, \tag{2.11}$$

and in which $\ll \cdot \gg$ denote the cumulants of η. Note that $\exp_o(\mathbf{F}(t))$ in Eq. (2.10) is an abbreviation in the sense that in order to evaluate this expression we must first expand it in powers of its argument observing the time ordering. For a Gaussian process all cumulants of order higher than 2 vanish and because $\ll \eta(t_1)\eta(t_2) \gg = \langle\eta(t_1)\eta(t_2)\rangle$ we obtain for this case:

$$\langle \mathbf{G}(t)\rangle = \exp_o\left(\frac{\varepsilon^2}{2!}\int_0^t\int_0^t dt_1 dt_2 \mathbf{H}(t_1)\mathbf{H}(t_2)\langle\eta(t_1)\eta(t_2)\rangle\right)$$
$$= \exp_o\left(\frac{\varepsilon^2}{2!}\int_0^t\int_0^t dt_1 dt_2 \mathbf{H}(t_1)\mathbf{H}(t_2)e^{-\nu(t_1-t_2)}\right). \tag{2.12}$$

Consequently, the ensemble average of the interaction perturbation obeys the equation:

$$\frac{d\langle\phi\rangle}{dt} = \left(\varepsilon^2\mathbf{H}(t)\int_0^t ds\mathbf{H}(s)e^{\nu(t-s)}\right)\langle\phi\rangle\,, \tag{2.13}$$

from which we obtain,

$$\frac{d\langle\psi\rangle}{dt} = \mathbf{A}\langle\psi\rangle + e^{\mathbf{A}t}\frac{d\langle\phi\rangle}{dt}$$
$$= \mathbf{A}\langle\psi\rangle + \varepsilon^2 e^{\mathbf{A}t}\mathbf{H}(t)d\int_0^t ds\mathbf{H}(s)e^{\nu(t-s)}e^{-\mathbf{A}t}\langle\psi\rangle\,, \tag{2.14}$$

giving the exact evolution equation for the ensemble average perturbation:

$$\frac{d\langle\psi\rangle}{dt} = \left(\mathbf{A} + \varepsilon^2\mathbf{B}\int_0^t e^{\mathbf{A}s}\mathbf{B}e^{-\mathbf{A}s}e^{-\nu s}ds\right)\langle\psi\rangle\,. \tag{2.15}$$

□

3. Approximate Equations for the Ensemble Mean

We seek approximations to the ensemble mean equation (2.2) that are valid for short autocorrelation time, $t_c = 1/\nu \ll 1$. The integrand defining the matrix $\mathbf{D}(t)$ given by Eq. (2.3) can be expanded as

$$e^{\mathbf{A}s}\mathbf{B}e^{-\mathbf{A}s}e^{-\nu s} = e^{-\nu s}\left(\mathbf{B} + s[\mathbf{A},\mathbf{B}] + \frac{s^2}{2!}[\mathbf{A},[\mathbf{A},\mathbf{B}]] + \cdots\right), \tag{3.16}$$

where $[\mathbf{A},\mathbf{B}] \equiv \mathbf{AB} - \mathbf{BA}$ is the commutator. The ensemble mean Eq. (2.2) then takes the exact form:

$$\frac{d\langle\psi\rangle}{dt} = \left[\mathbf{A} + \varepsilon^2\mathbf{B}\left(I_0\mathbf{B} + I_1[\mathbf{A},\mathbf{B}] + \frac{I_2}{2!}[\mathbf{A},[\mathbf{A},\mathbf{B}]] + \cdots\right)\right]\langle\psi\rangle\,, \tag{3.17}$$

where $I_n = \int_0^t s^n e^{-\nu s} ds$. Because $I_{n+1} = O(1/\nu^{n+1})$, for $1/\nu \ll 0$ the ensemble mean accurately evolves by retaining only the first power of $1/\nu$ in Eq. (3.17). In that limit the ensemble mean equation becomes:

$$\frac{d\langle\psi\rangle}{dt} = \left(\mathbf{A} + \frac{\varepsilon^2}{\nu}\mathbf{B}^2\right)\langle\psi\rangle \,. \tag{3.18}$$

This equation is identical to the ensemble mean equation obtained for $\eta(t)$ a white noise process, in which case as the limit $\nu \to \infty$ is taken the fluctuation amplitude also increases so that the ratio ε^2/ν approaches a constant; and Eq. (3.18) becomes the equation appropriate for a white noise process $\eta(t)$ with variance $2\varepsilon^2/\nu$ in the physically relevant Stratonovich limit ([1]).

4. Obtaining the Optimal Perturbations of Uncertain Systems

Definition 4.1. The optimal perturbation of an uncertain linear dynamical system for time t is the unit magnitude initial ($t = 0$) perturbation that maximizes the expected perturbation magnitude in the chosen norm at time t. The expected growth in magnitude of the optimal perturbation is called the optimal growth.

For deterministic dynamical systems governed by non-normal operators the optimal growth in the L_2 norm is the 2-norm of the fundamental matrix evolved to time t and the optimal perturbations can be readily found by a Schmidt decomposition (singular value decomposition) of the fundamental matrix at time t. We extend this result for obtaining the optimal perturbations to take account of the uncertainty in the dynamics as follows:

Theorem 4.1. *The optimal perturbation at time t of the uncertain dynamical system* (2.1) *is the eigenfunction associated with the largest eigenvalue of the Hermitian matrix $\langle\mathbf{S}(t)\rangle$ that is obtained by integrating to time t the differential equation*:

$$\frac{d\langle\mathbf{S}\rangle}{dt} = (\mathbf{A} + \varepsilon^2\mathbf{E}(t)\mathbf{B})^\dagger\langle\mathbf{S}\rangle + \langle\mathbf{S}\rangle(\mathbf{A} + \varepsilon^2\mathbf{E}(t)\mathbf{B}) + \varepsilon^2(\mathbf{E}(t)^\dagger\langle\mathbf{S}\rangle\mathbf{B} + \mathbf{B}^\dagger\langle\mathbf{S}\rangle\mathbf{E}(t)) \,, \tag{4.19}$$

where

$$\mathbf{E}(t) = \int_0^t e^{-\mathbf{A}t'}\mathbf{B}e^{\mathbf{A}t'}e^{-\nu t'}dt' \,, \tag{4.20}$$

with initial condition $\langle\mathbf{S}(0)\rangle = \mathbf{I}$.

Proof. Let $\Phi(t,0)$ be the fundamental matrix associated with each realization of the operator $\mathbf{A} + \varepsilon\eta(t)\mathbf{B}$. We seek the initial perturbation leading to greatest expected perturbation magnitude at future time, t. At time t the perturbation square amplitude for each realization of the fluctuations is:

$$\psi^\dagger(t)\psi(t) = \psi^\dagger(0)\Phi^\dagger(t,0)\Phi(t,0)\psi(0) \,, \tag{4.21}$$

where $\psi(0)$ the initial state. It is apparent from this expression that the eigenvector of the hermitian matrix:

$$\mathbf{S}(t) = \Phi^\dagger(t,0)\Phi(t,0) \tag{4.22}$$

with largest eigenvalue is the initial condition leading to greatest perturbation magnitude at time t for that realization of the fluctuations. The other eigenvectors of $\mathbf{S}(t)$ complete the set of mutually orthogonal initial conditions ordered according to their growth at time t.

The hermitian matrix $\mathbf{S}(t)$ can be determined by integrating forward the equation:

$$\frac{d\mathbf{S}}{dt} = (\mathbf{A} + \varepsilon\eta(t)\mathbf{B})^\dagger\mathbf{S} + \mathbf{S}(\mathbf{A} + \varepsilon\eta(t)\mathbf{B})\,. \tag{4.23}$$

The initial perturbation resulting in maximum mean square perturbation magnitude at time t is the eigenfunction corresponding to the largest eigenvalue of the mean of $\mathbf{S}(t)$. We need therefore to obtain the mean evolution equation corresponding to the matrix Eq. (4.23). This can be readily achieved using the results of the previous section by first expressing Eq. (4.23) as a vector equation using tensor products. This is done by associating with the $n \times n$ matrix $\mathbf{S}(t)$ the n^2 column vector $s(t)$ formed by consecutively stacking the columns of $\mathbf{S}(t)$. In tensor form Eq. (4.23) becomes:

$$\frac{ds}{dt} = \boldsymbol{\mathcal{A}}s + \varepsilon\eta(t)\boldsymbol{\mathcal{B}}s\,, \tag{4.24}$$

in which:

$$\boldsymbol{\mathcal{A}} = \mathbf{I}\otimes\mathbf{A}^\dagger + \mathbf{A}^{\mathrm{T}}\otimes\mathbf{I}\,, \qquad \boldsymbol{\mathcal{B}} = \mathbf{I}\otimes\mathbf{B}^\dagger + \mathbf{B}^{\mathrm{T}}\otimes\mathbf{I}\,, \tag{4.25}$$

where T denotes the transposed matrix and "†" the Hermitian conjugate matrix. The ensemble average evolution equation for s over Gaussian realizations of $\eta(t)$, indicated by $\langle s\rangle$, is obtained by applying Theorem 2.1 to (4.24) with result:

$$\frac{d\langle s\rangle}{dt} = (\boldsymbol{\mathcal{A}} + \varepsilon^2\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{D}}(t))\langle s\rangle\,, \tag{4.26}$$

where

$$\boldsymbol{\mathcal{D}}(t) = \int_0^t e^{\boldsymbol{\mathcal{A}}t'}\boldsymbol{\mathcal{B}}e^{-\boldsymbol{\mathcal{A}}t'}e^{-\nu t'}\,dt'\,. \tag{4.27}$$

Because $e^{\boldsymbol{\mathcal{A}}t} = e^{\mathbf{A}^{\mathrm{T}}t}\otimes e^{\mathbf{A}^\dagger t}$, repeated application of the tensor product properties gives

$$\boldsymbol{\mathcal{D}}(t) = \mathbf{I}\otimes\mathbf{E}(t)^\dagger + \mathbf{E}(t)^{\mathrm{T}}\otimes\mathbf{I}\,, \tag{4.28}$$

where

$$\mathbf{E}(t) = \int_0^t e^{-\mathbf{A}t'}\mathbf{B}e^{\mathbf{A}t'}e^{-\nu t'}\,dt'\,. \tag{4.29}$$

The ensemble average evolution equation can then be written as:

$$\frac{d\langle s\rangle}{dt} = \boldsymbol{\mathcal{L}}\langle s\rangle \tag{4.30}$$

in which $\boldsymbol{\mathcal{L}}$ is given by:

$$\boldsymbol{\mathcal{L}} = \mathbf{I}\otimes\mathbf{A}^{\dagger} + \mathbf{A}^{\mathrm{T}}\otimes\mathbf{I} + \varepsilon^2(\mathbf{I}\otimes\mathbf{B}^{\dagger} + \mathbf{B}^{\mathrm{T}}\otimes\mathbf{I})(\mathbf{I}\otimes\mathbf{E}(t)^{\dagger} + \mathbf{E}(t)^{\mathrm{T}}\otimes\mathbf{I})\,. \tag{4.31}$$

Reverting to matrix notation we obtain that the ensemble mean, $\langle\mathbf{S}\rangle$, obeys the deterministic equation:

$$\frac{d\langle\mathbf{S}\rangle}{dt} = (\mathbf{A} + \varepsilon^2\mathbf{E}(t)\mathbf{B})^{\dagger}\langle\mathbf{S}\rangle + \langle\mathbf{S}\rangle(\mathbf{A} + \varepsilon^2\mathbf{E}(t)\mathbf{B}) + \varepsilon^2(\mathbf{E}(t)^{\dagger}\langle\mathbf{S}\rangle\mathbf{B} + \mathbf{B}^{\dagger}\langle\mathbf{S}\rangle\mathbf{E}(t))\,. \tag{4.32}$$

□

For short autocorrelation times for the operator fluctuations corresponding to $1/\nu \ll 1$ the ensemble mean matrix $\langle S\rangle$ satisfies the white noise equation:

$$\frac{d\langle\mathbf{S}\rangle}{dt} = \left(\mathbf{A} + \frac{\varepsilon^2}{\nu}\mathbf{B}^2\right)^{\dagger}\langle\mathbf{S}\rangle + \langle\mathbf{S}\rangle\left(\mathbf{A} + \frac{\varepsilon^2}{\nu}\mathbf{B}^2\right) + \frac{2\varepsilon^2}{\nu}\mathbf{B}^{\dagger}\langle\mathbf{S}\rangle\mathbf{B}\,. \tag{4.33}$$

In their appropriate limits these equations can be used to obtain $\langle\mathbf{S}(t)\rangle$ and eigenanalysis of $\langle\mathbf{S}(t)\rangle$ in turn determines the optimal initial condition that leads to the largest expected growth in square magnitude at time t. Determining the optimal in this manner also offers constructive proof of the remarkable fact that the optimal initial covariance matrix has rank 1 implying that a sure initial condition maximizes expected growth in an uncertain system.

5. Conclusions

Uncertainty in perturbation dynamical systems can arise from many sources including statistical specification of parameters (Sardeshmukh *et al.* [12]; Palmer [10]) and incomplete knowledge of the mean state. We have obtained dynamical equations for ensemble mean and second moment quantities in such systems that are generally valid and others that are valid in the limit of short autocorrelation times. Optimal perturbation plays a central role in generalized stability analysis and we have used ensemble mean second moment equations to solve for the perturbation producing the greatest expected second moment growth in an uncertain system; remarkably, this optimal perturbation is sure.

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