Perturbation Growth and Structure in Uncertain Flows. Part I

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ABSTRACT

Perturbation growth in uncertain systems is examined and related to previous work in which linear stability concepts were generalized from a perspective based on the nonnormality of the underlying linear operator. In this previous work the linear operator, subject to an initial perturbation or a stochastic forcing distributed in time, was either fixed or time varying, but in either case the operator was certain. However, in forecast and climate studies, complete knowledge of the dynamical system being perturbed is generally lacking; nevertheless, it is often the case that statistical properties characterizing the variability of the dynamical system are known. In the present work generalized stability theory is extended to such uncertain systems. The limits in which fluctuations about the mean of the operator are correlated over time intervals, short and long, compared to the timescale of the mean operator are examined and compared with the physically important transitional case of operator fluctuation on timescales comparable to the timescales of the mean operator. Exact and asymptotically valid equations for transient ensemble mean and moment growth in uncertain systems are derived and solved. In addition, exact and asymptotically valid equations for the ensemble mean response of a stable uncertain system to deterministic forcing are derived and solved. The ensemble mean response of the forced stable uncertain system obtained from this analysis is interpreted under the ergodic assumption as equal to the time mean of the state of the uncertain system as recorded by an averaging instrument. Optimal perturbations are obtained for the ensemble mean of an uncertain system in the case of harmonic forcing. Finally, it is shown that the remarkable systematic increase in asymptotic growth rate with moment in uncertain systems occurs only in the context of the ensemble.

1. Introduction

Linear stability theory for fluid systems has been extensively studied because of its role in advancing understanding of physical phenomena, including structure and growth of perturbations, growth of errors in forecast models, transition from laminar to turbulent flow, and maintenance of the turbulent state. Historically, linear stability theory has addressed problems of deterministic growth using the method of modes (Rayleigh 1880; Charney 1947; Eady 1949). However, the method of modes is incomplete for understanding perturbation growth even for autonomous systems because the nonnormality of the linear operator in physical problems often produces transient development of a subset of perturbations that dominates the physically relevant growth processes (Kelvin 1887; Farrell 1982, hereafter F82). Recognition of the role of nonnormality in linear sta-

bility led to the development of generalized stability theory (GST; F82; Farrell 1988; Farrell and Ioannou 1996a, hereafter FI96a). Compared to the methods of modes, the methods of GST, which are based on the nonnormality of the linear operator, allow a far wider class of stability problems to be addressed, including perturbation growth associated with aperiodic time-dependent certain operators to which the method of modes does not apply (Farrell and Ioannou 1996b, hereafter FI96b; Farrell and Ioannou 1999, hereafter FI99). An example of such an aperiodic time-dependent stability problem is the forecast error growth problem in which the nonnormality of a certain but time-dependent linear system, the tangent linear operator of the forecast, produces asymptotic Lyapunov instability (FI99). In addition to addressing problems of deterministic growth of sure initial perturbations, GST also addresses problems of the growth of statistical distributions of perturbations to certain operators allowing prediction of statistical quantities including variance, fluxes, and structures in turbulent flow (Farrell and Ioannou 1993a,b), which has led to a new mechanistic model of geophys-

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ical and laboratory turbulence (Farrell and Ioannou 1994, 1995, 1998).

There remains a class of linear stability problems still to be addressed by the methods of GST, which is the stability of uncertain systems. The problems described above involve growth of perturbations in a system with no time dependence or a system with known time dependence: the perturbations to this system may be sure or stochastically distributed and may be imposed at the initial time, or distributed continuously in time, but the operator to which the forcing is applied is considered to be certain. However, it may happen that we do not have complete knowledge of the system that is being perturbed. For instance, the parameterizations of damping and radiation in the tangent linear forecast equations may not be certain but may rather include realistic statistical variability about their mean values. Another example is departures from the climatological mean of realizations of the planetary-scale Northern Hemisphere flow. These realizations are drawn from an ensemble of statistically similar flows but, as is well known, there are significant temporal variations of the planetary waves about the mean. In these examples the system governing the growth and structure of perturbations depends not only on the mean state but also on uncertain but statistically known variations of the state about its mean. Three regimes in the statistical analysis of uncertain system are distinguished: systems in which the time dependence of the statistical fluctuations of the operator are temporally correlated for short intervals compared to the damping and oscillation timescales of the associated mean operator, systems in which the statistical fluctuations of the mean operator are correlated for long time intervals compared to the timescales of the mean operator, and the physically important transitional case of operator fluctuation on timescales comparable to those of the mean operator.

Describing the stability of uncertain systems requires introducing the distinct concepts of sample stability, moment stability, and ensemble mean stability (Arnold 1984; Arnold and Kliemann 1987). Remarkably, uncertain systems that are sample (Lyapunov) stable may be higher-order moment (e.g., variance) unstable and uncertain systems that are sample and moment unstable may be ensemble mean stable. We show that a single realization of a time-dependent system exhibits only the sample (Lyapunov) asymptotic exponential growth for all its moments.

Compared to growth of ensemble mean quantities themselves, growth of ensemble mean second-moment quantities corresponds more closely to notions of stability and transient growth as generally understood. The sample (Lyapunov) growth rate is attained asymptotically by any single realization of the second moment, yet ensemble second-moment quantities can be unstable even for sample stable systems. We interpret these results physically using analysis based on the ensemble probability density function.

Ensemble mean and moment quantities are physically interpretable as applying to single realizations when the associated operator is stable and forced so that the ergodic assumption linking the ensemble and time mean can be made, in which case the ensemble mean and moment quantities apply to observations made using a time-averaging instrument. In Part I of this paper we show that in the ensemble mean of an uncertain system there is an optimal structure producing the greatest response to harmonic forcing and so the ensemble mean frequency response as well as the structure of the optimal forcing and response in the ensemble mean at each frequency apply to time averages of a sufficiently long time series from a single realization, if the ergodic assumption is valid.

In Farrell and Ioannou (2002, hereafter Part II) growth and stability of second-moment quantities are examined in more detail and the optimal perturbation problem for initial and stochastic forcing is cast and solved in the context of the second moment. We illustrate methods for analyzing ensemble mean growth in uncertain systems using simple examples including barotropic and baroclinic flow models with fluctuating winds and fluctuating damping parameters.

2. Moment stability in uncertain flows

The familiar concept of sample (Lyapunov) instability generalizes to aperiodic certain systems the asymptotic exponential modal instability of autonomous systems and the asymptotic Floquet instability of periodic systems. The Lyapunov exponent of an ensemble of state vector solutions $\psi(t)$ produced by realizations of an uncertain system is defined as

$$\lambda_0 = \left\langle \lim_{t \to \infty} \frac{\ln|\psi(t)|}{t} \right\rangle, \tag{1}$$

in which the angle bracket refers to an average over a sufficiently large ensemble of members, $\psi(t)$, produced by statistically similar realizations of a dynamical system. It is also useful (cf. Arnold 1984; Arnold and Kliemann 1987) to consider the asymptotic ensemble mean exponent of the first moment of the state vector,

$$\lambda_1 = \lim_{t \to \infty} \frac{\ln\langle|\psi(t)|\rangle}{t}; \tag{2}$$

the asymptotic ensemble mean exponent of the second moment,

$$\lambda_2 = \frac{1}{2} \lim_{t \to \infty} \frac{\ln\langle|\psi(t)|^2\rangle}{t}; \tag{3}$$

and, in general,

$$\lambda_p = \frac{1}{p} \lim_{t \to \infty} \frac{\ln\langle|\psi(t)|^p\rangle}{t}. \tag{4}$$

These stability measures are not generally distinguished

because the Lyapunov and moment exponents are the same for certain systems. Moreover, these stability measures are accompanied by characteristic structures that are also identical for certain systems. However, in the case of uncertain systems, although every individual realization is more and more likely with increasing time to approach the Lyapunov growth and structure for all moments, the ensemble mean exponents are asymptotically unequal and ordered as

$$\lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \quad (5)$$

and associated with each of the exponents is a distinct structure.

For example, consider a system in which growth, g, takes the value of 2 or ½, with equal probability over unit time intervals. The Lyapunov exponent is

$$\lambda_0 = \langle \ln g \rangle = \frac{\ln(2) + \ln(1/2)}{2} = 0, \quad (6)$$

while the first- and second-moment exponents are

$$\lambda_1 = \ln\langle g \rangle = \ln\left(\frac{2 + 1/2}{2}\right) = \ln\left(\frac{5}{4}\right) = 0.22, \quad (7)$$

$$\lambda_2 = \frac{1}{2}\ln\langle g^2 \rangle$$

$$= \frac{1}{2}\ln\left[\frac{2^2 + (1/2)^2}{2}\right] = \frac{1}{2}\ln\left(\frac{17}{8}\right) = 0.38, \quad (8)$$

and in general the p-moment exponent, as defined, is given by

$$\lambda_p = \frac{1}{p}\ln\langle g^p \rangle = \frac{1}{p}\ln\left[\frac{2^p + (1/2)^p}{2}\right], \quad (9)$$

with limit[1]

$$\lim_{p \to \infty} \lambda_p = \ln 2 = 0.69. \quad (10)$$

We can use probability density functions of the ensemble member growth rates to examine moment growth more closely. If the growth rate of a perturbation is calculated after n units of time there are $n + 1$ possible growth rates distributed on the interval $[\log(1/2), \log(2)] = [-0.69, 0.69]$ according to the binomial distribution (Fig. 1). As n increases the distribution of growth rates approaches a δ function about the Lyapunov exponent, $\lambda_0 = 0$. However, despite the concentration of the distribution around $\lambda = 0$, for every n the growth rates remain distributed over the interval $[-0.69, 0.69]$, and despite the vanishing probability as $n \to \infty$ of any given realization achieving a growth rate significantly different from λ_0, the higher-moment exponents remain sufficiently influenced by the finite extent of this binomial

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[1] That the limit of $p \to 0$ is the Lyapunov exponent λ_0 has been shown in Arnold (1984) and Arnold and Kliemann (1987).

distribution to give for all n the nonzero growth rates obtained in (9) (right panel of Fig. 1).

The above example illustrates that a deeper understanding of moment instability can be obtained from the probability density function (pdf) of the growth rates of the uncertain system as a function of time, assuming that all perturbations start from a sure initial condition. Consider that the pdf, $P_t(\lambda)$, of growth rates over an interval t,

$$\lambda(t) = \frac{\ln(|\psi|)}{t}, \quad (11)$$

has been determined. The Lyapunov exponent over this interval is the mean of this distribution,

$$\lambda_0(t) = \int_{-\infty}^{\infty} \lambda P_t(\lambda) \, d\lambda, \quad (12)$$

while the first p moment exponents are

$$\lambda_p(t) = \frac{1}{pt}\ln\left[\int_{-\infty}^{\infty} e^{pt\lambda}P_t(\lambda) \, d\lambda\right]. \quad (13)$$

That a growth rate pdf of finite width results in the ordering of the higher-moment exponents $\lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$ is clear from (13). In contrast to the finite higher-moment exponents of the binomial pdf (Fig. 1), Gaussian pdf's inevitably lead to instability of sufficiently high moments because the Gaussian can produce ensemble members, however rare, with unbounded growth rates that dominate the higher-moment exponent obtained in (13) (Arnold and Kliemann 1987). The pdf's of physical quantities are often approximately Gaussian near their mean but with vanishing probability of extreme values so that a distribution similar to the binomial may be more appropriate than the Gaussian in predicting higher-moment growth for physical systems. Moreover, for practical purposes the distinction between the Lyapunov exponent and moment exponents fades as the integration interval increases because it becomes increasingly unlikely that any realization deviates from the Lyapunov exponent and structure.

The fact that as time increases the width of the distribution of growth rates narrows and that in the limit $t \to \infty$ the pdf becomes an infinitely narrow distribution about the Lyapunov exponent is known as the multiplicative ergodic theorem of Oseledec (Arnold 1998); this allows calculation of the Lyapunov exponent of an uncertain system from a single sufficiently long integration of the perturbation equations. On the other hand, as we have seen, the growth rate of higher-order moments depends on the finite extent of the growth rate pdf even as this pdf becomes increasingly centered about the Lyapunov exponent. If, for example, the distribution of growth rates $P_t(\lambda)$ is a Gaussian of width $\delta(t) = \{\langle\lambda^2(t)\rangle\}^{1/2}$, then according to (13) the Lyapunov exponent of the p moment over the time interval t is
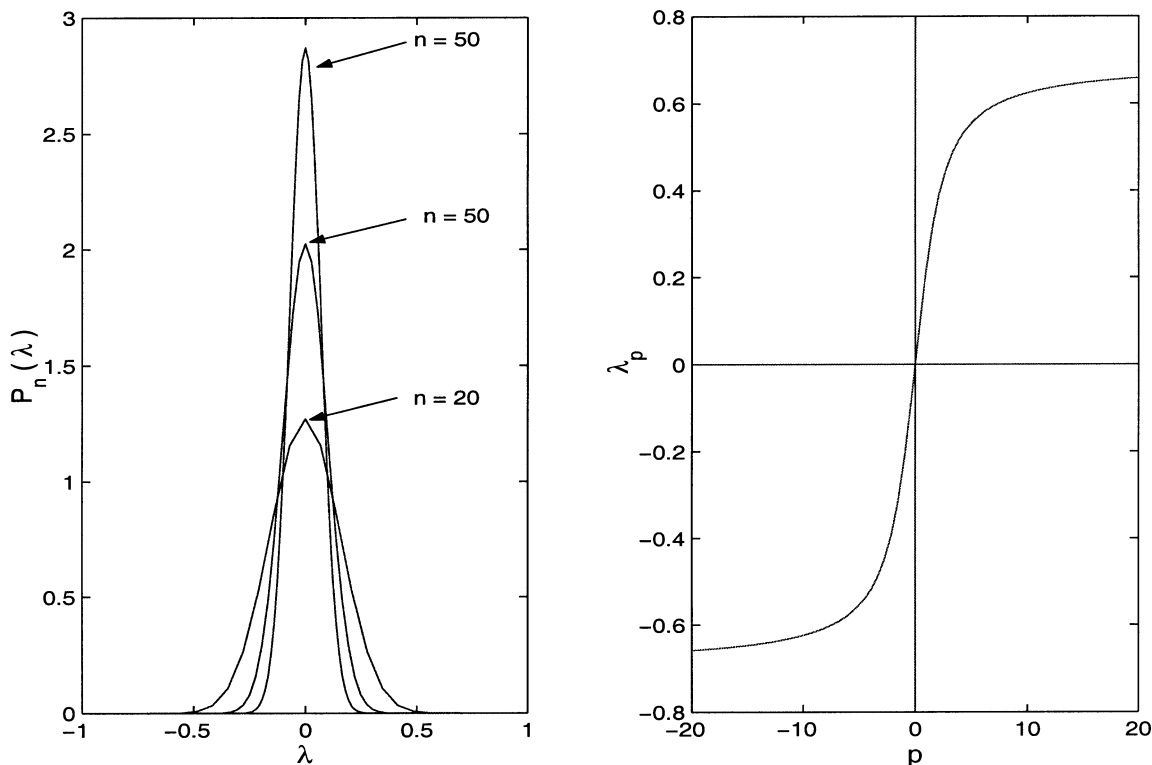
FIG. 1. (left) Binomial probability distribution function, $P_n(\lambda)$, of the growth rate λ after n unit time steps for the process in which at each time step the perturbation either grows by 2 or decays by 1/2. The distributions are centered about the Lyapunov exponent, which is equal to $\lambda_0 = 0$. Shown are the pdf's for $n = 20$, 50, 100. The distributions become increasingly narrow, assuming a Gaussian form. A large number of trials is needed in order to obtain growth rates at the wings of the distribution. These rare events determine the exponents of higher-order moments. (right) Moment exponents as a function of moment order. The $p = 0$ moment exponent is the Lyapunov exponent. The higher-order moment exponents asymptote to 0.69, unlike in cases of exactly Gaussian statistics for which the higher-moment exponent increases linearly with moment order. The moment exponent curve for the binomial distribution is independent of n.

$$\lambda_p(t) = \frac{1}{pt} \ln(\langle e^{p\lambda t} \rangle). \quad (14)$$

A closed form for the expected value of the growth factor $e^{p\lambda t}$ in (14) is obtained by a cumulant expansion using the fact that the higher-order cumulants of a Gaussian distribution vanish (Van Kampen 1992):

$$\langle e^{pt\lambda} \rangle = \exp\left\{pt\left[\langle\lambda(t)\rangle + \frac{pt}{2}\delta^2(t)\right]\right\}. \quad (15)$$

The p-moment exponent is then given exactly by

$$\lambda_p(t) = \langle\lambda(t)\rangle + \frac{pt}{2}\delta^2(t), \quad (16)$$

from which the $t \to \infty$ asymptotic dependence of $\lambda_p(\infty)$ on the width of the pdf, $\delta(t)$, is clear. If we assume that $\delta^2(t) = O(1/t)$, as when the growth rate distribution is Gaussian, then asymptotically the p-moment Lyapunov exponent increases linearly in the moment p, confirming that regardless of the value of the Lyapunov exponent, $\lambda_0 = \lim_{t\to\infty}\langle\lambda(t)\rangle$, sufficiently high-order moments will be unstable.

In order to illustrate these ideas consider the example

of the modified Eady problem in which the zonal flow varies about a constant mean shear. The model is Boussinesq with constant stratification on an f plane and has periodic boundary conditions in the zonal, x, direction; solid walls in the meridional, y, direction; and a solid lid at height $z = H$. The zonal flow $U(z, t, \omega)$ is generalized to be time dependent and uncertain, a realization of which, ω, is taken from a probability space Ω. Horizontal scales are nondimensionalized by $L = 1200$ km, vertical scales by $H = fL/N = 10$ km, velocity by $U_0 = 50$ m s^{-1}, and time by $T = L/U_0$, so that a time unit is approximately 6.7 h. The Brunt–Väisälä frequency is $N = 10^{-2}$ s^{-1}, and the Coriolis parameter is $f = 10^{-4}$ s^{-1}.

The nondimensional linearized equation that governs evolution of streamfunction perturbations ψ for each realization ω is

$$\frac{\partial D^2\psi}{\partial t} = -ikU(z, t, \omega)D^2\psi$$

$$+ ik\frac{d^2U(z, t, \omega)}{dz^2}\psi - rD^2\psi, \quad (17)$$

in which the perturbation is assumed to be of form $\psi(z,$

t, ω) $e^{ikx+ily}$, where k is the zonal and l is the meridional wavenumber: the perturbation potential vorticity is $D^2\psi$, with $D^2 \equiv \partial^2/\partial z^2 - k^2 - l^2$ and the perturbation potential vorticity damping rate is r.

Conservation of potential temperature at the ground, $z = 0$, and tropopause, $z = 1$, provides the boundary conditions

$$\frac{\partial^2\psi}{\partial t\partial z} = -ikU(0, t, \omega)\frac{\partial\psi}{\partial z} + ikU'(0, t, \omega)\psi$$

$$- r\frac{\partial\psi}{\partial z} + \Gamma_g(k^2 + l^2)\psi \quad \text{at } z = 0, \quad (18)$$

$$\frac{\partial^2\phi}{\partial t\partial z} = -ikU(1, t, \omega)\frac{\partial\psi}{\partial z} + ikU'(1, t, \omega)\psi - r\frac{\partial\psi}{\partial z}$$

$$\text{at } z = 1, \quad (19)$$

where $U'(0, t, \omega)$ and $U'(1, t, \omega)$ denote the velocity shear at $z = 0$ and $z = 1$, respectively. The coefficient of Ekman damping $\Gamma_g \equiv N/U_0(\nu/2f)^{1/2}$ has value $\Gamma_g = 0.0632$, corresponding to a vertical eddy momentum diffusion coefficient $\nu = 20$ m^2 s^{-1} in the boundary layer.

The zonal wind $U(z, t, \omega) = z + u(z, t, \omega)$ is composed of an ensemble mean component, z, and uncertain deviations, $u(z, t, \omega)$. The linear perturbation dynamics is

$$\frac{d\psi}{dt} = [\mathbf{A} + \mathbf{A}_1(t, \omega)]\psi, \quad (20)$$

where $\mathbf{A}$ is the sure Eady operator,

$$\mathbf{A} = (D^2)^{-1}(-ikzD^2 - rD^2), \quad (21)$$

and $\mathbf{A}_1(t, \omega)$ is the uncertain operator associated with the wind fluctuations,

$$\mathbf{A}_1(t, \omega)$$

$$= (D^2)^{-1}\left[-iku(z, t, \omega)D^2 + ik\frac{d^2u(z, t, \omega)}{dz^2}\right]. \quad (22)$$

In both operators, D^2 has been rendered invertible by incorporating the boundary conditions.

Assume further that the uncertain zonal flow has form $u(z, t, \omega) = \epsilon \sum_i \xi_i(t, \omega)u_i(z)$, in which the wind fluctuations result from modulation of a set of wind profiles, $u_i(z)$, weighted by the identically distributed random variable, $\xi_i(t, \omega)$, and ϵ is an amplitude factor. In terms of the time-independent operators $\mathbf{B}_i$ the uncertain operator has the form

$$\mathbf{A}_1(t, \omega) = \epsilon \sum_i \xi_i(t, \omega)\mathbf{B}_i, \quad (23)$$

where each $\mathbf{B}_i$ is the certain operator,

$$\mathbf{B}_i = (D^2)^{-1}\left[-iku_i(z)D^2 + ik\frac{d^2u_i(z)}{dz^2}\right]. \quad (24)$$

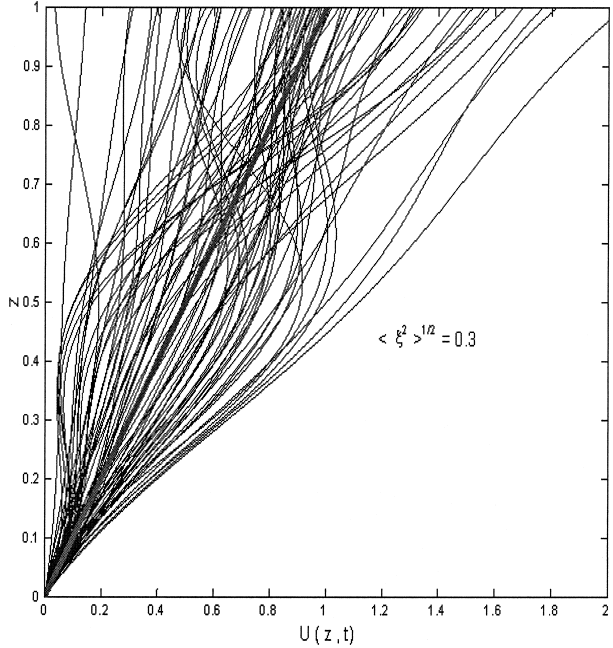For the examples the uncertain wind is formed by the superposition of four wind profiles:

FIG. 2. Realizations of the time-dependent velocity $U(z, t) = z + u(z, t)$. The bold line is the mean wind, $U(z) = z$.

$$u_1(z) = \frac{1}{2}\left[1 - \cos\left(\frac{\pi z}{2}\right)\right], \quad u_2(z) = \frac{1}{2}[1 - \cos(2\pi z)],$$

$$u_3(z) = \sin\left(\frac{\pi z}{2}\right), \quad \text{and} \quad u_4(z) = z^2; \quad (25)$$

and the random functions $\eta_i(t, \omega)$ are generated by the stochastic differential equation,

$$d\eta_i = -\nu\eta_i dt + dW, \quad (26)$$

with W a Wiener process. In (26) the symbol η has been used in place of ξ as an indication that the random variable is Gaussian. The standard deviation of η_i is $(2\nu)^{-1/2}$ and the autocorrelation time is $1/\nu$. The wind profiles were chosen to give variance increasing with height, and also to almost always produce westerly winds. Typical realizations of the resulting mean flow with $(\langle\xi_i^2\rangle)^{1/2} = 0.3$ and autocorrelation time $t_c = 1$ are shown in Fig. 2.

Consider growth rates over the relatively short interval 8.5 days with meridional wavenumber $l = 2$, noise autocorrelation time 7 h, and an rms fluctuation amplitude $\epsilon = 0.3$. Moment growth rates as a function of wavenumber are shown in Fig. 3. The middle curve is the ensemble mean 8.5-day growth rate as a function of wavenumber with the bars indicating the standard deviation from the ensemble mean. Individual realizations result in widely different growth rates over this time interval. The top curve shows the exponent of the second moment, λ_2, and demonstrates the ordering $\lambda_0 \leq \lambda_2$. For reference the optimal growth rate for the mean flow over the same interval is also shown. Because of
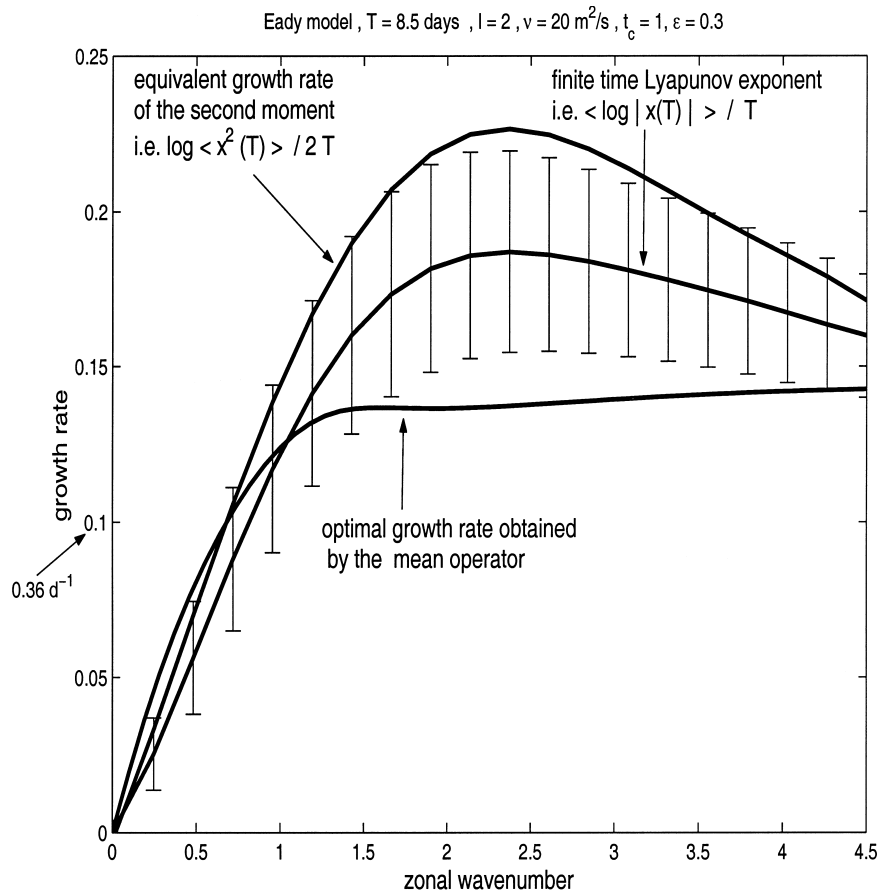
FIG. 3. Growth rates over 8.5 days as a function of wavenumber in the Eady model with uncertain fluctuating winds. The middle curve is the 8.5-day ensemble mean growth rate and the bars indicate one std dev from the ensemble mean value. The top curve shows the exponent of the second moment, which is by necessity larger than the mean growth rate. For reference the growth rate attained by the 8.5-day optimal perturbation on the mean flow is also shown. The time dependence of the flow leads to an increase of growth rate at almost all wavenumbers. The meridional wavenumber is $l = 2$, the coefficient of Ekman damping corresponds to a vertical diffusion coefficient of $\nu = 20$ m² s⁻¹, the autocorrelation time of the noise is 7 h, and the rms of the fluctuating components is 0.3.

the time dependence of the flow, the ensemble mean growth rate has increased for almost all wavenumbers.

The pdf of the growth rate obtained from 10^5 simulations over $t = 80$ days is shown in Fig. 4. This time interval was chosen to be long enough so that initial conditions do not influence the result, and the mean of the pdf is a good approximation of the asymptotic Lyapunov exponent. In this example the Lyapunov exponent is negative but the second and higher moments are unstable, as shown in Fig. 5. The large p growth rates shown are not the asymptotic values because the limited number of simulations in this Monte Carlo calculation do not explore the unlikely events responsible for producing the higher-moment growth rates. If the number of simulations is increased, the higher-order moments asymptote to higher values, and in the limit of an infinite number of simulations the moment exponents grow linearly with p, as required by (16) given Gaussian wind

fluctuations. However, as previously mentioned, the Gaussian pdf overestimates the probability of rare events in many physical problems so this asymptote is not likely to be observed.

Consider an ensemble of flows that are Lyapunov (sample) stable while at the same time being second-moment unstable. Such an ensemble would have divergent energy and variance although the ensemble mean magnitude of the perturbations decays to zero. This is because relatively rare high growth trajectories dominate the second moment (e.g., positive growth rates in the pdf shown in Fig. 4). Associated with these high growth rate realizations are structures that differ from the structures associated with the sample mean, which grows at the rate of the top Lyapunov exponent and has the structure of the first Lyapunov vector. The most unstable second-moment structure is the top EOF, which is calculated from the covari-
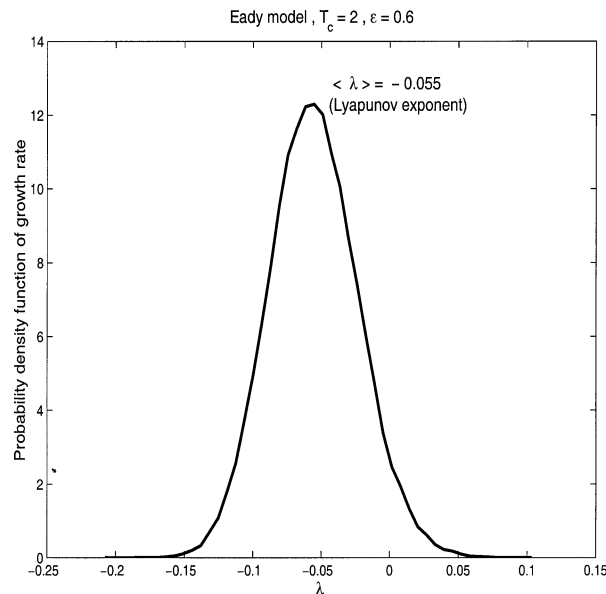
FIG. 4. Pdf of the growth rate of perturbations over 80 days in the Eady model with fluctuating winds. The autocorrelation time of the fluctuations is $t_c = 2$ and their rms amplitude is $\epsilon = 0.6$. The Lyapunov exponent is $\langle \lambda \rangle = -0.055$; nevertheless, this flow is second-moment unstable with second-moment exponent $\langle \lambda_2 \rangle = 0.05$.

ance matrix; EOFs for uncertain systems are examined in Part II of this paper.

3. Dynamics of the ensemble mean field

Consider a statistical description of perturbation growth in which the perturbation dynamics is governed by an uncertain linear operator with realizations $\mathbf{A}(t, \omega)$, where ω is taken from a probability space Ω. We concentrate on uncertain operators $\mathbf{A}(t, \omega)$ of the form $\mathbf{A}(t, \omega) = \mathbf{A} + \epsilon \eta(t, \omega)\mathbf{B}$, with $\eta(t, \omega)$ a Gaussian process[2] with zero mean, unit variance, and specified autocorrelation time t_c.

We have seen that the stability properties of an uncertain system can be understood from the pdf of the growth rate. For example, because the width of the growth rate pdf becomes vanishingly small with time the Lyapunov exponent and related structure can be obtained almost surely from analysis of a single long integration of the perturbation equations. Therefore, analysis of the structure of the first Lyapunov vector does not require consideration of ensembles (cf. FI99). We turn now to the problem of analyzing ensemble average growth and growth of higher-order moments. If we start with a sure initial condition in an (additively) unforced but multiplicatively uncertain system, a difference among the higher-order moment growth rates and the Lyapunov growth rate exists only in an ensemble of flow realizations and it seems at first that our study

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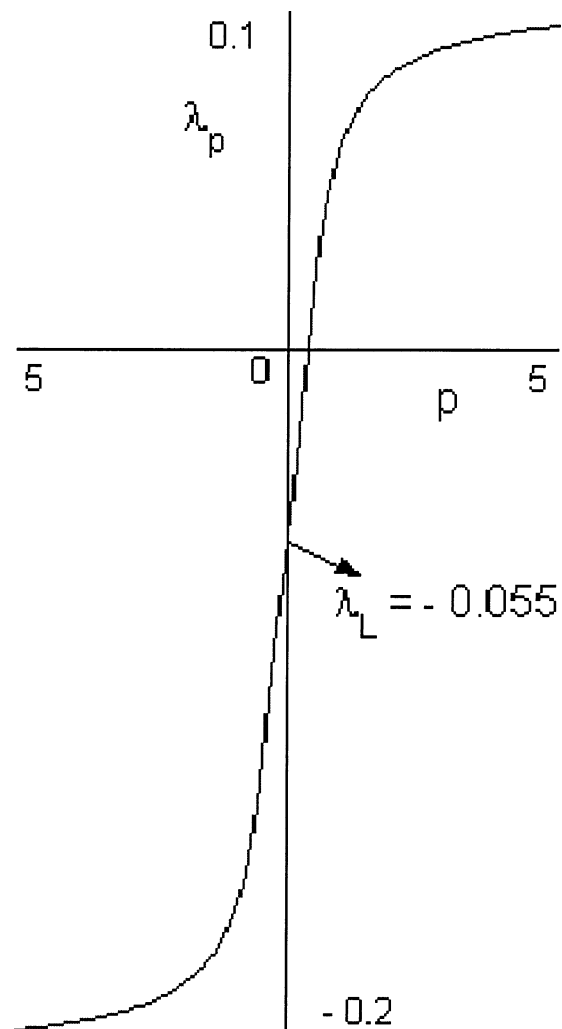[2] The ω will henceforth be omitted for brevity.

FIG. 5. Exponents of higher moments, λ_p, as a function of moment order, p. These exponents were obtained using the pdf shown in Fig. 4.

requires Monte Carlo simulation. However, this computationally expensive task can in many cases be avoided, because deterministic equations exist governing the evolution of higher-order moments in uncertain systems under appropriate restrictions.

An equation for the evolution of the ensemble average field, $\langle \psi(t) \rangle$, and the second moment, $\langle \psi_i(t)\psi_j^*(t) \rangle$, can be readily obtained if the temporal variability of the fluctuations is taken to be white in time (Arnold 1992). Although it is a great advantage for analysis to take the temporal randomness in the uncertain operators to be white, this assumption is often hard to justify because white noise processes in common with Gaussian processes have unbounded fluctuations, for example, implying infinite wind speeds or negative damping rates. We wish to obtain equations that are not restricted in validity to the white noise limit, at least in part to assess the validity of the white noise limit.

We begin with derivation of an equation for the ensemble mean. While ensemble mean stability is not directly related to moment stability, it has a distinct and interesting interpretation that we will discuss, and it underpins the development of equations for the second moment. Uncertain second-moment dynamics, which is the subject of Part II of this paper, is of particular interest because it is associated with perturbation energy and the EOFs of the stochastically forced uncertain system, and its stability properties correspond more clearly to usual notions of perturbation growth.

We first derive the ensemble mean equation valid for the case in which the time mean operator $\mathbf{A}$ commutes with the perturbation operator $\mathbf{B}$, so that the commutator $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ vanishes. We then deal with the case in which the commutator $[\mathbf{A}, \mathbf{B}]$ is nonzero.

Dynamics of the ensemble mean field for commuting mean and fluctuation operators

Consider the model linear system

$$\frac{d\psi}{dt} = \mathbf{A}\psi + \epsilon\eta(t)\mathbf{B}\psi \qquad (27)$$

in which the uncertain operator is assumed to take the special form $\mathbf{A}(t) = \mathbf{A} + \epsilon\eta(t)\mathbf{B}$, and in which the operators $\mathbf{A}$ and $\mathbf{B}$ are assumed to commute. Because they commute, the operators $\mathbf{A}$ and $\mathbf{B}$ can be simultaneously diagonalized by a change of coordinates and the evolution of the perturbations can be followed in these normal coordinates.[3] The behavior of this n-dimensional system in normal coordinates is that of n uncoupled one-dimensional uncertain systems. The exact equation for the mean can be obtained if $\eta(t)$ is a Gaussian process with zero mean and unit variance that decorrelates with time at rate ν, that is, $\langle\eta(t_1)\eta(t_2)\rangle = \exp(-\nu|t_1 - t_2|)$. For such a process the autocorrelation time, t_c, is $t_c = 1/\nu$. Because $[\mathbf{A}, \mathbf{B}] = 0$, the propagator from t to s is

$$\mathbf{P}(t, s) = \exp\left[\mathbf{A}(t - s) + \epsilon\mathbf{B}\int_s^t \eta(s)\, ds\right], \qquad (28)$$

and the statistical properties of the propagator are determined by $\theta(t, s) \equiv \int_s^t \eta(s)\, ds$. If the pdf of θ, $p[\theta(t, s)]$, is known then the mean of the propagators is

$$\langle\mathbf{P}(t, s)\rangle = e^{\mathbf{A}(t-s)}\int_{-\infty}^{\infty} d\theta\, p[\theta(t, s)]e^{\epsilon\mathbf{B}\theta(t, s)}. \qquad (29)$$

The pdf, $p[\theta(t, s)]$, can be obtained for all times because the central limit theorem guarantees that the integral, $\theta(t, s)$, of the zero mean Gaussian processes, $\eta(t)$, will also be distributed Gaussian with pdf:

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[3] This use of the term normal refers to the orthogonality of the modes of $\mathbf{A}$ in these coordinates and is unrelated to Gaussian normal distributions.

$$p[\theta(t, s)] = \sqrt{\frac{1}{2\pi\langle\theta^2(t, s)\rangle}}\exp\left(-\frac{\theta^2}{2\langle\theta^2(t, s)\rangle}\right), \qquad (30)$$

In the present case the variance is

$$\langle\theta^2(t, s)\rangle = \int_s^t\int_s^t\langle\eta(s_1)\eta(s_2)\rangle\, ds_1\, ds_2$$
$$= 2\frac{t - s}{\nu} - 2\frac{1 - e^{-\nu(t-s)}}{\nu^2}, \qquad (31)$$

so from (29) the mean of the propagators becomes

$$\langle\mathbf{P}(t, s)\rangle = \sqrt{\frac{1}{2\pi\langle\theta^2(t, s)\rangle}}e^{\mathbf{A}(t-s)}$$
$$\times \int_{-\infty}^{\infty} d\theta \exp\left(-\frac{\theta^2}{2\langle\theta^2(t, s)\rangle}\mathbf{I} + \epsilon\mathbf{B}\theta\right)$$
$$= \exp\left(\mathbf{A}(t - s) + \epsilon^2\frac{\langle\theta^2(t, s)\rangle}{2}\mathbf{B}^2\right)$$
$$= \exp\left[\mathbf{A}(t - s) + \frac{\epsilon^2}{\nu}\mathbf{B}^2\left(t - s - \frac{1 - e^{-\nu(t-s)}}{\nu}\right)\right], \qquad (32)$$

where $\mathbf{I}$ is the identity. It can be easily verified using (32) that the mean of the propagators, $\langle\mathbf{P}(t, s)\rangle$, does not satisfy the semigroup property $\langle\mathbf{P}(t, 0)\rangle = \langle\mathbf{P}(t, s)\rangle\langle\mathbf{P}(s, 0)\rangle$, implying that the mean field at time t resulting from a sure initial condition at time s, $\langle\psi(t)\rangle = \langle\mathbf{P}(t, s)\rangle\psi(s)$, satisfies a differential equation depending explicitly on the initial time s. In the particular case of evolution from $t = 0$ the mean field $\langle\psi\rangle \equiv \langle\psi(t, 0)\rangle$ satisfies

$$\frac{d\langle\psi\rangle}{dt} = \left[\mathbf{A} + \frac{\epsilon^2}{\nu}(1 - e^{-\nu t})\mathbf{B}^2\right]\langle\psi\rangle. \qquad (33)$$

It is important to bear in mind this restriction to evolution from $t = 0$ on this useful equation, which is exact in commuting case with Gaussian noise.

In the limit of very small autocorrelation times ($\nu \to \infty$) for all $t \gg 1/\nu$ we obtain asymptotically the ensemble mean equation

$$\frac{d\langle\psi\rangle}{dt} = \left(\mathbf{A} + \frac{\epsilon^2}{\nu}\mathbf{B}^2\right)\langle\psi\rangle, \qquad (34)$$

which is without explicit dependence on the initial time. The associated mean propagator is

$$\langle\mathbf{P}(t)\rangle = \exp\left(\mathbf{A} + \frac{\epsilon^2}{\nu}\mathbf{B}^2\right)t. \qquad (35)$$

Equation (34) is familiar as it governs the ensemble mean of (27) in the white noise limit for arbitrary $\mathbf{A}$ and $\mathbf{B}$ (Arnold 1992; Sardeshmukh et al. 2001).

From (33) it is clear that for all autocorrelation times and all noise levels the asymptotic mean behavior of

the uncertain system follows (34). Because this is the same ensemble mean behavior that would be obtained if the noise were white with variance $2\epsilon^2/\nu$, the operator on the rhs of (34) will be referred to as the equivalent white noise operator. Equation (34) shows that asymptotically the effect of the fluctuations on the ensemble mean field is to modify the mean propagator by a term proportional to $\mathbf{B}^2$, which for imaginary eigenvalues of $\mathbf{B}$ contributes an additional damping to that of the mean operator, $\mathbf{A}$. This extra damping results from phase averaging over the realizations of the ensemble. In contrast, real eigenvalues of $\mathbf{B}$ increase the growth over that produced by the mean operator. This increased asymptotic growth results from the convexity of the exponential function in a manner analyzed in the context of time-dependent stability by FI99.

While (33) is exact for Gaussian distributed $\theta(t)$, the restriction on validity of (33) to Gaussian pdf's limits its utility for fluctuations with long autocorrelation times, $t_c = 1/\nu$. We interpret this restriction as follows: each realization of the propagator (28) (with $s = 0$) contains a factor $e^{\epsilon\theta(t)\mathbf{B}}$ in which $\theta(t) \equiv \int_0^t \eta(s)\,ds$ for $\eta(t)$ Gaussian with standard deviation 1. Consider replacing the Gaussian distribution η by a restricted distribution of fluctuations $\xi(t)$ that is distributed Gaussian up to fluctuation magnitude Ξ_0, with zero probability of obtaining values $|\xi(t)| > \Xi_0$. Then for each such realization the term in the exponent, $\epsilon\theta(t)$, will be at most $\epsilon\Xi_0 t$; and when the averaging is performed the mean exponent must be smaller than $\epsilon\Xi_0 t$ so that if restricted pdf's are to be modeled as Gaussian distributions it must be required according to (32) that $\epsilon\Xi_0 t > \epsilon^2 t/\nu$ for $t \gg t_c$, which in turn requires $\Xi_0 > \epsilon/\nu = \epsilon t_c = K$. The nondimensional number $K = \epsilon t_c$ is called the Kubo number. We conclude that (33) is accurate for $\xi(t)$ with restricted pdf only for Kubo numbers that do not exceed the maximum magnitude of the fluctuation, here taken to be Ξ_0. The Kubo number can be interpreted heuristically as the range in standard deviations over which rms fluctuations need to be distributed Gaussian for (33) to hold. Physical quantities such as wind or damping fluctuations seldom retain Gaussian statistics much beyond two standard deviations, so $K = 2$ is a practical upper bound for application of (33) to physical problems.

For fluctuations with autocorrelation times sufficiently long that the associated Kubo number exceeds that for which physical fluctuations can be accurately modeled as Gaussian the ensemble mean equation (33) ceases to be valid. For such cases an approximate solution can be obtained by restricting the fluctuations to two states: a plus state for which the propagator is $\mathbf{A} + \epsilon\mathbf{B}$ and a minus state for which the propagator is $\mathbf{A} - \epsilon\mathbf{B}$. If the probability of residence in each state decays exponentially with time as e^{-t/t_c}, then an exact expression can be obtained for the evolution of the ensemble mean propagator that is valid even for noncommuting $\mathbf{A}$ and $\mathbf{B}$. In appendix A it is shown that the ensemble means,

$\langle\psi(t)\rangle_\pm$, which denote the ensemble mean on the condition that at time t the fluctuation is in the $\pm$ state, obey the equations

$$\frac{d\langle\psi\rangle_+}{dt} = (\mathbf{A} + \epsilon\mathbf{B})\langle\psi\rangle_+ + \frac{\nu}{2}\langle\psi\rangle_- - \frac{\nu}{2}\langle\psi\rangle_+, \quad (36a)$$

$$\frac{d\langle\psi\rangle_-}{dt} = (\mathbf{A} - \epsilon\mathbf{B})\langle\psi\rangle_- + \frac{\nu}{2}\langle\psi\rangle_+ - \frac{\nu}{2}\langle\psi\rangle_-, \quad (36b)$$

from which the total ensemble mean field $\langle\psi(t)\rangle = \langle\psi(t)\rangle_+ + \langle\psi(t)\rangle_-$ is obtained. It is clear from (36a) and (36b) that in the limit of long correlation times, that is, for $\nu = 1/t_c \to 0$, the propagator of the ensemble mean becomes the average propagator of the two states without dependence on the correlation time. This prediction is in stark disagreement with the Gaussian ensemble mean propagator (35), which contains the mean fluctuation parameter ϵ^2/ν that dominates the dynamics even for arbitrarily small ϵ if fluctuations correlate for a sufficiently long interval (i.e., $\nu \to 0$). We see that this behavior occurs because of the generally unphysical unbounded fluctuations permitted by a Gaussian pdf.

4. Ensemble mean dynamics for quasi-static fluctuations

Assume that as the autocorrelation time $t_c = 1/\nu \to \infty$ the rms amplitude of the operator fluctuation, ϵ, tends to a finite nonzero value. In such cases the propagator can be approximated by its quasi-static limit in which the time dependence of the operator $\mathbf{A} + \epsilon\xi(t)\mathbf{B}$ is neglected but not its randomness. The ensemble mean propagator in the quasi-static approximation is consequently the propagator averaged over the fluctuation realizations

$$\langle\mathbf{P}(t)\rangle = \langle e^{[(\mathbf{A}+\epsilon\xi\mathbf{B})t]}\rangle$$

$$= \int_{-\infty}^{\infty} d\xi\, P(\xi)e^{[(\mathbf{A}+\epsilon\xi\mathbf{B})t]}, \quad (37)$$

where the average is over the probability distribution of the fluctuations, $P(\xi)$. The quasi-static approximation is formally valid for $t \ll t_c$. However, its validity extends for all times if all realizations of $\mathbf{A} + \epsilon\xi(t)\mathbf{B}$ lead to perturbation decay, with decay times shorter than t_c. Note that (37) is also valid whether or not $\mathbf{A}$ and $\mathbf{B}$ commute and is not limited to Gaussian fluctuations.

For fluctuations, η, which are Gaussian with zero mean and unit variance we replace ξ by η in (37) to obtain the mean propagator

$$\langle\mathbf{P}(t)\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{(\mathbf{A}+\epsilon\eta\mathbf{B})t}e^{-\eta^2/2}\,d\eta. \quad (38)$$

Alternatively, in the frequency domain, as the resolvent of each realization is

$$\mathbf{R}(\omega) = \int_0^\infty e^{i\omega t} e^{(\mathbf{A}t + \epsilon\xi\mathbf{B}t)} \, dt$$

$$= (-i\omega\mathbf{I} - \mathbf{A} - \epsilon\xi\mathbf{B})^{-1}, \qquad (39)$$

the resolvent corresponding to the quasi-static response is the mean resolvent over the probability distribution, $P(\xi)$:

$$\mathbf{R}_s(\omega) = \int_{-\infty}^\infty d\xi P(\xi)(-i\omega\mathbf{I} - \mathbf{A} - \epsilon\xi\mathbf{B})^{-1}, \quad (40)$$

assuming that all realizations of the operator $\mathbf{A} - \epsilon\xi\mathbf{B}$ are stable.

5. A barotropic channel example in which the mean and fluctuation operator commute

Consider meridionally independent zonal flow fluctuations about a constant zero shear zonal channel flow (cf. Sardeshmukh et al. 2001). Streamfunction perturbations $\psi(t)e^{ikx}$, where x is the coordinate in the zonal direction, are governed by the barotropic vorticity equation

$$\frac{d\psi}{dt} = -ik[U_0 + \epsilon\eta(t)]\psi - r\psi, \qquad (41)$$

where U_0 is the mean flow, r is the damping rate, ϵ is the amplitude of the wind fluctuations, and the flow fluctuations $\eta(t)$ are assumed to be Gaussian with autocorrelation time $t_c = 1/\nu$. In this case the mean operator $\mathbf{A} = -ikU_0 - r$ and the fluctuation operator $\mathbf{B} = -ik$ commute and the ensemble mean equation (33) is

$$\frac{d\langle\psi\rangle}{dt} = \left[-ikU_0 - r - k^2\frac{\epsilon^2}{\nu}(1 - e^{-\nu t})\right]\langle\psi\rangle, \quad (42)$$

with associated propagator

$$\langle\mathbf{P}(t)\rangle = \exp\left[-ikU_0 t - rt - k^2\frac{\epsilon^2}{\nu}\left(t - \frac{1 - e^{-\nu t}}{\nu}\right)\right]. \quad (43)$$

For times $t \ll t_c$, (43) predicts Gaussian decay,

$$\langle\mathbf{P}(t)\rangle_0 = \exp\left(-ikU_0 t - rt - k^2\frac{\epsilon^2}{2}t^2\right), \quad (44)$$

while for $t \gg t_c$, (43) reduces to the white noise approximation,

$$\langle\mathbf{P}(t)\rangle = \exp\left(-ikU_0 t - rt - k^2\frac{\epsilon^2}{\nu}t\right). \quad (45)$$

Although each realization of ψ decays at the damping rate r independent of wavenumber, (45) predicts that the ensemble mean, $\langle\psi\rangle$, obtained by averaging over all realizations has an additional rate of decay that is asymptotically proportional to the square of the wavenumber, as if the ensemble were subject to a diffusion

with coefficient ϵ^2/ν. This effective diffusion results from cancellation among the realizations and is not a property of any single realization. This effective diffusion is a general feature of vacillating advection because the fluctuation operator $\mathbf{B}$ is of the form $\mathbf{u} \cdot \nabla$, where $\mathbf{u}$ is the fluctuating wind, and consequently the $\mathbf{B}^2$ operator in (34) has the form of an anisotropic diffusion $\mathbf{u} \cdot \nabla(\mathbf{u} \cdot \nabla)$ (cf. Knobloch 1978).

Consider a mean flow with $U_0 = 5$, $r = 0$, and perturbations with wavenumber $k = 1/5$. The mean of 2000 realizations of (41) with these parameters is shown in Fig. 6. The solution found using the exact propagator (43) is indistinguishable from this mean. Remarkably, even for autocorrelation time $t_c = 3$ and fluctuation amplitude $\epsilon = 0.3$ ($K = 0.9$) the asymptotic ensemble mean propagator (45) accurately captures the decay of the ensemble mean for all times (top panel). An example of the evolution of a sure initial condition for a larger Kubo number ($K = 2$) is shown in Fig. 6 (bottom panel); in this case the equivalent white noise prediction (45) overestimates the decay.

The Kubo number appropriate for wind fluctuations in this model is estimated from the expression $K = ku't_c$, where u' is the amplitude of the wind fluctuations, and k the perturbation wavenumber. A 30–m s^{-1} mean flow speed and a 1000-km perturbation spatial scale result in an advective timescale of 0.4 days. Flow fluctuations with 4-day autocorrelation and rms speed 1/10 that of the mean flow correspond to nondimensional autocorrelation time $t_c = 10$ and nondimensional fluctuation amplitude $\epsilon = 1/10$, which gives Kubo number $K = 1$. Because we expect an approximate Gaussian pdf for wind fluctuations up to one standard deviation with $K = 1$, the Gaussian approximations is valid.

Consider now the effect of fluctuation of the damping coefficient. In forecast and climate models damping parameters are uncertain and it has been argued that stochastic variation of damping parameters may have a beneficial effect on simulations (Palmer 2001; Sardeshmukh et al. 2001). Variation of damping does not change the oscillation period and phase cancellation does not lead to decay of the ensemble average in this case. Fluctuations of damping do modify the decay rate of each realization and with the fluctuation operator $\mathbf{B} = 1$ the exact propagator obtained from (32) is

$$\langle\mathbf{P}(t)\rangle = \exp\left[-ikU_0 t - rt + \frac{\epsilon^2}{\nu}\left(t - \frac{1 - e^{-\nu t}}{\nu}\right)\right], \quad (46)$$

the limit of which for $t \ll t_c$ is

$$\langle\mathbf{P}(t)\rangle_0 = \exp\left(-ikU_0 t - rt + \frac{\epsilon^2}{2}t^2\right), \quad (47)$$

while for $t \gg t_c$ the limit is

$$\langle\mathbf{P}(t)\rangle_\infty = \exp\left(-ikU_0 t - rt + \frac{\epsilon^2}{\nu}t\right). \quad (48)$$

Variation of damping results in a mean perturbation exceeding the perturbation obtained using the mean operator. For sufficiently large fluctuations the ensemble is destabilized because for some realizations the damping is negative, giving rise to growth rather than decay. Negative damping is unrealistic, and the implied instability of the ensemble mean is unphysical.

6. Ergodic interpretation of the ensemble mean

The ensemble mean can be straightforwardly interpreted as the result of averaging over a large number of realizations as in taking the mean of an ensemble of forecasts. Alternatively, although the ensemble mean is not defined for a single realization, it is of theoretical and practical interest to determine under what circum-

stances the ensemble mean can be interpreted as applying to a single realization.

We make the ergodic assumption that if the system reaches a stationary state then its time mean is the same as its ensemble mean. This assumption has been verified using Monte Carlo simulations for uncertain systems excited by both additive deterministic and stochastic forcing. Note that identifying the ensemble mean with the time mean requires that the system be forced so that a nontrivial stationary state is obtained.

Consider the harmonically forced uncertain system

$$\frac{d\psi}{dt} = [\mathbf{A} + \epsilon\eta(t)\mathbf{B}]\psi + Fe^{-i\omega t}. \qquad (49)$$

Under the ergodic assumption the time mean of $\psi(t)e^{-i\omega t}$, $\psi(t)e^{i\omega t}$, which can be obtained by an averaging instrument, is

$$\overline{\psi(t)e^{i\omega t}} = \lim_{t\to\infty} \langle\psi(t)e^{i\omega t}\rangle = \lim_{t\to\infty} \int_0^t \langle \mathbf{P}(t, s)\rangle Fe^{i\omega(t-s)} \, ds$$

$$= \lim_{t\to\infty} \int_0^t \exp\left[\mathbf{A}(t - s) + \frac{\epsilon^2}{\nu}\mathbf{B}^2\left(t - s - \frac{1 - e^{-\nu(t-s)}}{\nu}\right)\right] Fe^{i\omega(t-s)} \, ds$$

$$= \int_0^\infty \exp\left[\mathbf{A}\tau + \frac{\epsilon^2}{\nu}\mathbf{B}^2\left(\tau - \frac{1 - e^{-\nu t}}{\nu}\right)\right] Fe^{i\omega\tau} \, d\tau$$

$$= \exp\left(-\frac{\epsilon^2\mathbf{B}^2}{\nu^2}\right) \int_0^\infty \exp\left(\mathbf{A}\tau + \frac{\epsilon^2}{\nu}\mathbf{B}^2\tau + i\omega\mathbf{I}\tau\right) \exp\left(\frac{\epsilon^2\mathbf{B}^2}{\nu^2}\epsilon^{-\nu\tau}\right) F \, d\tau$$

$$= \exp\left(-\frac{\epsilon^2\mathbf{B}^2}{\nu^2}\right) \int_0^\infty d\tau \exp\left(\mathbf{A}\tau + \frac{\epsilon^2}{\nu}\mathbf{B}^2\tau + i\omega\mathbf{I}\tau\right) \sum_{n=0}^\infty \frac{\epsilon^{2n}}{\nu^{2n}n!}\mathbf{B}^{2n}e^{-n\nu\tau} F$$

$$= \exp\left(-\frac{\epsilon^2\mathbf{B}^2}{\nu^2}\right) \sum_{n=0}^\infty \frac{\epsilon^{2n}}{\nu^{2n}n!}\mathbf{B}^{2n}\left[-\left(\mathbf{A} - n\nu\mathbf{I} + \frac{\epsilon^2}{\nu}\mathbf{B}^2\right) - i\omega\mathbf{I}\right]^{-1} F \equiv \mathbf{R}(\omega)F. \qquad (50)$$

where $\mathbf{I}$ is the identity. To arrive at the above expression we used (32) and assumed that the integrals exist, which requires that the operator $\mathbf{A} + \epsilon^2/\nu\mathbf{B}^2$ be asymptotically stable. The ensemble average resolvent $\mathbf{R}(\omega)$ that maps the sinusoidal forcing at frequency ω to the ensemble average response at this frequency is then identified as

$$\mathbf{R}(\omega) = \exp\left(-\frac{\epsilon^2\mathbf{B}^2}{\nu^2}\right) \sum_{n=0}^\infty \frac{\epsilon^{2n}}{\nu^{2n}n!}\mathbf{B}^{2n}$$

$$\times \left[-\left(\mathbf{A} - n\nu\mathbf{I} + \frac{\epsilon^2}{\nu}\mathbf{B}^2\right) - i\omega\mathbf{I}\right]^{-1}. \quad (51)$$

In this way the ensemble mean resolvent can be obtained from a time average over a single realization. In particular, the response to stationary forcing in a fluctuating

environment is given by the value of the resolvent at $\omega = 0$. The above results require that $\mathbf{A}$ and $\mathbf{B}$ commute.

Consider the example with commuting $\mathbf{A}$ and $\mathbf{B}$ in the fluctuating barotropic flow (41) forced sinusoidally at frequency ω. If station measurements are made and Fourier analyzed to obtain the frequency response, then with fluctuations a weakening of the response and a broadening of the spectrum occurs, in accord with the theory presented above, with the observed resolvent in the case of a fluctuating wind being

$$\mathbf{R}(\omega) = \exp\left(\frac{\epsilon^2 k^2}{\nu^2}\right) \sum_{n=0}^\infty (-1)^n\frac{(\epsilon k)^{2n}}{\nu^{2n}n!}$$

$$\times \frac{1}{-i\omega + ikU_0 + r + \nu n + \frac{\epsilon^2}{\nu}k^2}, \quad (52a)$$
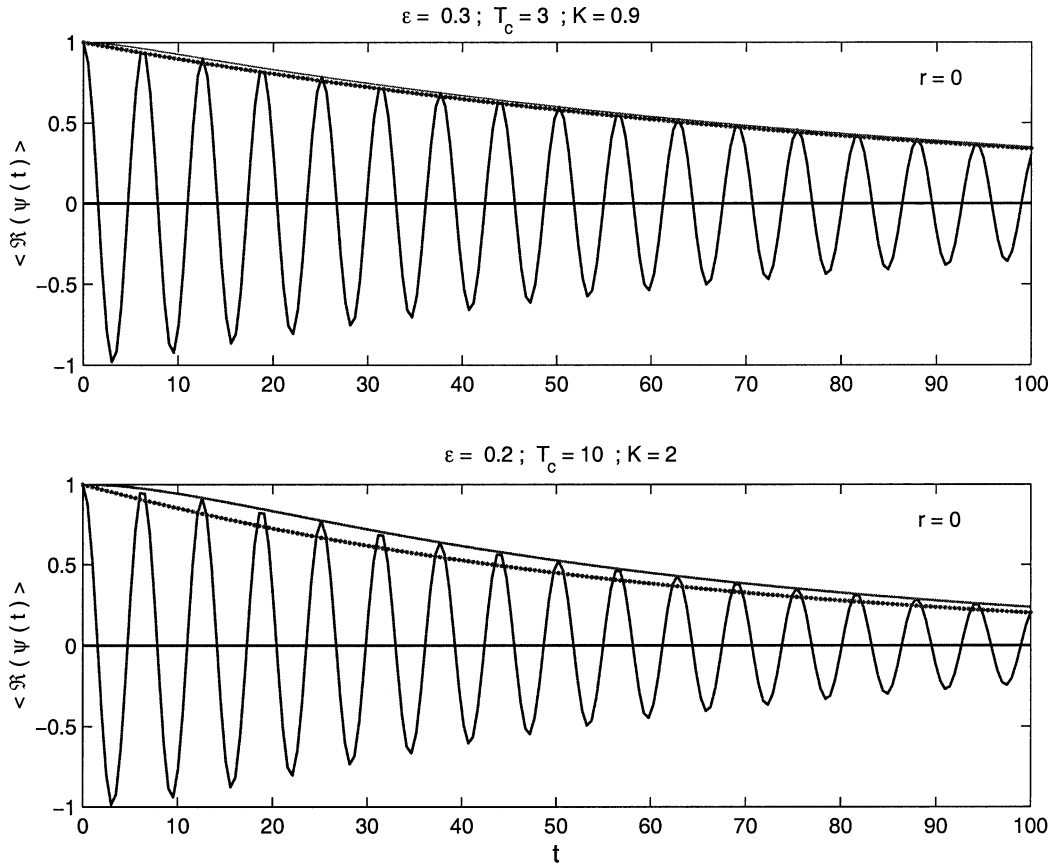
FIG. 6. The real part of $\langle\psi\rangle$ obtained from a 2000-member ensemble of simulations of (41). Also plotted is the curve identical to that predicted from the exact propagator for the commuting case (43). The amplitude $|\langle\psi\rangle|$ according to (43) and the amplitude according to the equivalent white noise approximation (45) are indicated with the dotted line: (top) with autocorrelation time $t_c = 3$ and fluctuation amplitude $\epsilon = 0.3$ ($K = 0.9$); the evolution follows closely the predictions of the white noise approximation; (bottom) with autocorrelation time $t_c = 10$ and fluctuation amplitude $\epsilon = 0.2$ ($K = 2$); the equivalent white noise predictions (dot curve) overestimates the decay rate. The mean operator is $\mathbf{A} = -i$ (the damping is $r = 0$) and the fluctuation operator is $\mathbf{B} = -i/5$.

$$\mathbf{R}(\omega) = \frac{1}{-i\omega + ikU_0 + r + \dfrac{\epsilon^2}{\nu}k^2}, \quad (52b)$$

with (52a) valid for general fluctuations, and (52b) valid in the short autocorrelation limit. These expressions should be compared with the resolvent

$$\mathbf{R}(\omega) = \frac{1}{-i\omega + ikU_0 + r}, \quad (53)$$

which is obtained in the absence of fluctuations.

The frequency response (52) is shown in Fig. 7 for fluctuation magnitude $\epsilon = 1/3$ and two autocorrelation times, (left panel) $t_c = 1/3$ ($K = 1/9$) and (right panel) $t_c = 6$ ($K = 2$). For reference the response with no fluctuations (53), and the quasi-static response (40) are also shown. For short autocorrelation times the response is close to the mean response, while for $t_c = 6$ ($K = 2$) the response is almost identical to the quasi static. The exact frequency response in (52) has been derived on

the assumption that the fluctuations are Gaussian, and it is not expected to describe in practice fluctuations which for large autocorrelation times have large corresponding Kubo numbers. As the Kubo number increases with increasing autocorrelation times the validity of (52) requires that the fluctuations retain their Gaussianity over a range of standard deviations equal to the Kubo number, K, something that is not expected in practice. For such cases the ensemble average frequency response is better approximated by the quasi-static approximation or by modeling the fluctuations by a telegraph Markov process as in appendix A.

In multidimensional systems the resolvent matrix contains information about the system's response to the spatial structure of the forcing. If the forcing structure is F, as in (49), then under the ergodic assumption the response at frequency, ω, as obtained by an averaging instrument, is $\mathbf{R}(\omega)F$, where $\mathbf{R}(\omega)$ is the resolvent of the ensemble mean equation. The resolvent matrix contains full information about the response of the system
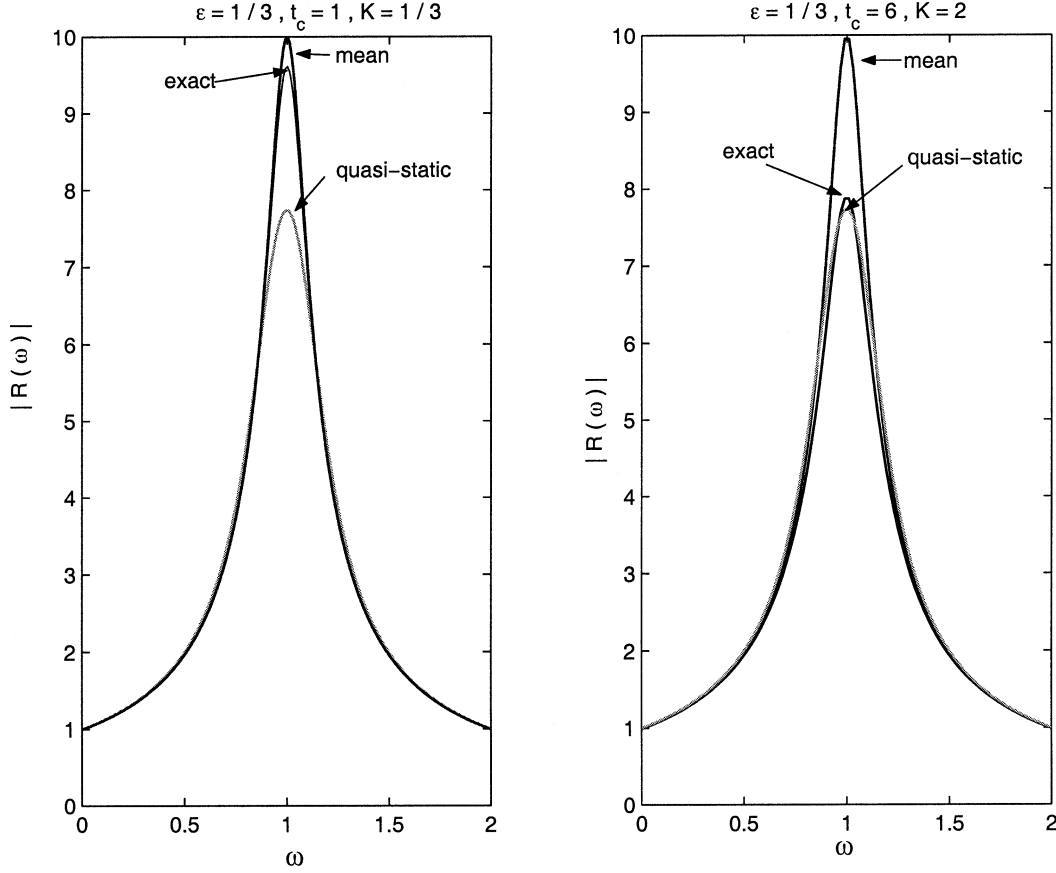
FIG. 7. Magnitude of the resolvent of the ensemble mean field as a function of frequency ω, in the barotropic flow example with fluctuating wind with rms amplitude $\epsilon = 1/3$. Also shown are the quasi-static response (40) that gives the response for autocorrelation times that are very large, the response in the absence of fluctuations (53), and the exact response (52): (left) for autocorrelation time $t_c = 1$; (right) for autocorrelation time $t_c = 6$ in which case the exact response is very close to the quasi-static response. The mean wind is $U_0 = 5$, the wavenumber is $k = 1/5$, and the coefficient of Ekman damping is $r = 0.1$. The fluctuation operator is $\mathbf{B} = -i/5$.

to any forcing structure and at any frequency but this may be more information than we need and the maximum singular value of the resolvent and its associated right singular vector may be sufficient information. The right singular vector of $\mathbf{R}(\omega)$ is the optimal forcing structure leading to the maximum response at frequency ω as recorded by an averaging instrument.

It should be noted that in uncertain systems the frequency response recorded by an averaging instrument is not the rms frequency response. The rms response, the square of which is the energy spectrum, is obtained from the resolvent of the second moment.

7. Ensemble mean dynamics for fluctuation operators that do not commute with the mean operator

Consider the case of noncommuting $\mathbf{A}$ and $\mathbf{B}$ in the model system

$$\frac{d\psi}{dt} = \mathbf{A}\psi + \epsilon\eta(t)\mathbf{B}\psi, \qquad (54)$$

with $\eta(t)$ a Gaussian process with zero mean and unit variance that decorrelates exponentially in time at rate ν.

The analysis is simplified by defining the interaction perturbation $\phi(t)$ by $\psi(t) \equiv e^{\mathbf{A}t}\phi(t)$. The interaction perturbation obeys the equation

$$\frac{d\phi}{dt} = \epsilon\eta(t)e^{-\mathbf{A}t}\mathbf{B}e^{\mathbf{A}t}\phi \equiv \epsilon\eta(t)\mathbf{H}(t)\phi. \qquad (55)$$

For sure initial perturbation $\phi(0) = \psi(0)$, the interaction perturbation at time t is

$$\phi(t) = \mathbf{G}(t)\psi(0), \qquad (56)$$

where the propagator $\mathbf{G}(t)$ is the time-ordered exponential

$$\mathbf{G}(t) = \exp_o\left[\epsilon \int_0^t \eta(s)\mathbf{H}(s)\ ds\right], \qquad (57)$$

in which the subscript o denotes time ordering; that is, in the series expansion of the exponential the terms are grouped in ascending time order. Because

$$\psi(t) = e^{\mathbf{A}t} \exp_o\left[\epsilon \int_0^t \eta(s)\mathbf{H}(s)\, ds\right]\psi(0), \quad (58)$$

the ensemble mean evolution is obtained by evaluating

$$\langle\mathbf{G}(t)\rangle = \left\langle\exp_o\left[\epsilon \int_0^t \eta(s)\mathbf{H}(s)\, ds\right]\right\rangle. \quad (59)$$

It was proven by Kubo (1962) that for Gaussian $\eta(t)$ this averaged propagator is exactly

$$\langle\mathbf{G}(t)\rangle = \exp_o\left[\epsilon^2 \int_0^t ds_1\mathbf{H}(s_1) \int_0^{s_1} ds_2\mathbf{H}(s_2)\langle\eta(s_1)\eta(s_2)\rangle\right]$$

$$= \exp_o\left[\epsilon^2 \int_0^t ds_1\mathbf{H}(s_1) \int_0^{s_1} ds_2\mathbf{H}(s_2)e^{-\nu(s_1-s_2)}\right]. \quad (60)$$

Consequently, the ensemble average of the interaction perturbation obeys the equation

$$\frac{d\langle\phi\rangle}{dt} = \left[\epsilon^2\mathbf{H}(t)\int_0^t ds\mathbf{H}(s)e^{-\nu(t-s)}\right]\langle\phi\rangle, \quad (61)$$

from which we obtain

$$\frac{d\langle\psi\rangle}{dt} = \mathbf{A}\langle\psi\rangle + e^{\mathbf{A}t}\frac{d\langle\phi\rangle}{dt}$$

$$= \mathbf{A}\langle\psi\rangle + \epsilon^2 e^{\mathbf{A}t}\mathbf{H}(t)\int_0^t ds\mathbf{H}(s)e^{-\nu(t-s)}e^{-\mathbf{A}t}\langle\psi\rangle, \quad (62)$$

giving the equation for evolution of the ensemble average perturbation starting at $t = 0$ that is valid for all Kubo numbers:

$$\frac{d\langle\psi\rangle}{dt} = \left(\mathbf{A} + \epsilon^2\mathbf{B}\int_0^t e^{\mathbf{A}s}\mathbf{B}e^{-\mathbf{A}s}e^{-\nu s}\, ds\right)\langle\psi\rangle. \quad (63)$$

This equation is exact if the noise is Gaussian. If $\mathbf{A}$ and $\mathbf{B}$ commute, we recover (33). The asymptotic stability of (63) is discussed in appendix B.

With sufficiently short autocorrelation times for the operator fluctuations, specifically for autocorrelation times $t_c \ll t_d$, where $t_d = \|\mathbf{A}\|^{-1}$ is the decay time of the most persistent mode of the mean operator, we obtain from (63) the ensemble equation valid for general $\mathbf{A}$ and $\mathbf{B}$:

$$\frac{d\langle\psi\rangle}{dt} = \left(\mathbf{A} + \frac{\epsilon^2}{\nu}\mathbf{B}^2\right)\langle\psi\rangle. \quad (64)$$

The ensemble mean equation (64) is also obtained in the limit of small Kubo number by perturbation techniques for non-Gaussian noise (Bourret 1962; Keller 1962; Papanicolaou and Keller 1971; Brissaud and Frisch 1974; see also Frisch 1968; Van Kampen 1974; Van Kampen 1992).

The white noise limit of (64) is obtained if the ratio ϵ^2/ν approaches a constant D, as $\epsilon \to \infty$ and $\nu \to \infty$. The rms of the white noise process is then $e_w = (2D)^{1/2}$ and the augmented operator governing evolution of the ensemble mean is $\mathbf{A} + e_w^2/2\mathbf{B}^2$, which recovers the Stratonovich correction for white noise–induced drift in the Fokker–Planck equation (Arnold 1992).

Although computationally convenient, the white noise limit is not physically realistic, because it implies unbounded fluctuations. It is remarkable that the equivalent white noise approximation (64), with a different interpretation of $D = \epsilon^2/\nu$, is also valid for non–white noise processes with sufficiently short autocorrelation times. In examples even operators with a 2-day autocorrelation time are described well by (64).

If there is more than one independent structure so that an instantaneous realization of the total operator is $\mathbf{A} + \Sigma_i\ \epsilon\xi_i(t)\mathbf{B}_i$, with independent random $\xi_i(t)$ having zero mean and unit variance, then the augmented deterministic operator that governs the evolution of the ensemble mean in (64) becomes $\mathbf{A} + \Sigma_i\ \epsilon_i^2/\nu\mathbf{B}_i^2$.

Examples of ensemble mean dynamics for fluctuation operators that do not commute with the mean operator

We have seen an example in which barotropic wind fluctuations lead to an effective diffusion of the ensemble mean. Diffusion is not the only result of operator fluctuation in physical systems, and an especially interesting example of a different effect is provided by the harmonic oscillator with a fluctuating spring constant. Let the harmonic oscillator's position x and velocity v be governed by the equation

$$\frac{d}{dt}\begin{pmatrix}x\\v\end{pmatrix} = \left[\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} + \epsilon\eta(t)\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\right]\begin{pmatrix}x\\v\end{pmatrix}, \quad (65)$$

in which the mean operator $\mathbf{A}$ and deviation operator $\mathbf{B}$ are easily identified. Note that because $\mathbf{B}^2 = 0$, in the short autocorrelation time limit the mean position evolves as if there were no fluctuation in the spring constant [cf. (64)] and the ensemble mean equations are

$$\frac{d}{dt}\begin{pmatrix}\langle x\rangle\\\langle v\rangle\end{pmatrix} = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}\langle x\rangle\\\langle v\rangle\end{pmatrix}. \quad (66)$$

This result is remarkable given that it is well known that the randomly modulated harmonic oscillator has a positive mean Lyapunov exponent despite the fact that at each time instant the linearized operator is neutrally stable (FI96b).

This property of the dynamics of the harmonic oscillator may be better appreciated by examining Fig. 8, where it can be seen that the ensemble average of 500 realizations follows quite accurately the ensemble mean equation (66). This result is obtained only for short autocorrelation times for the spring constant fluctuations
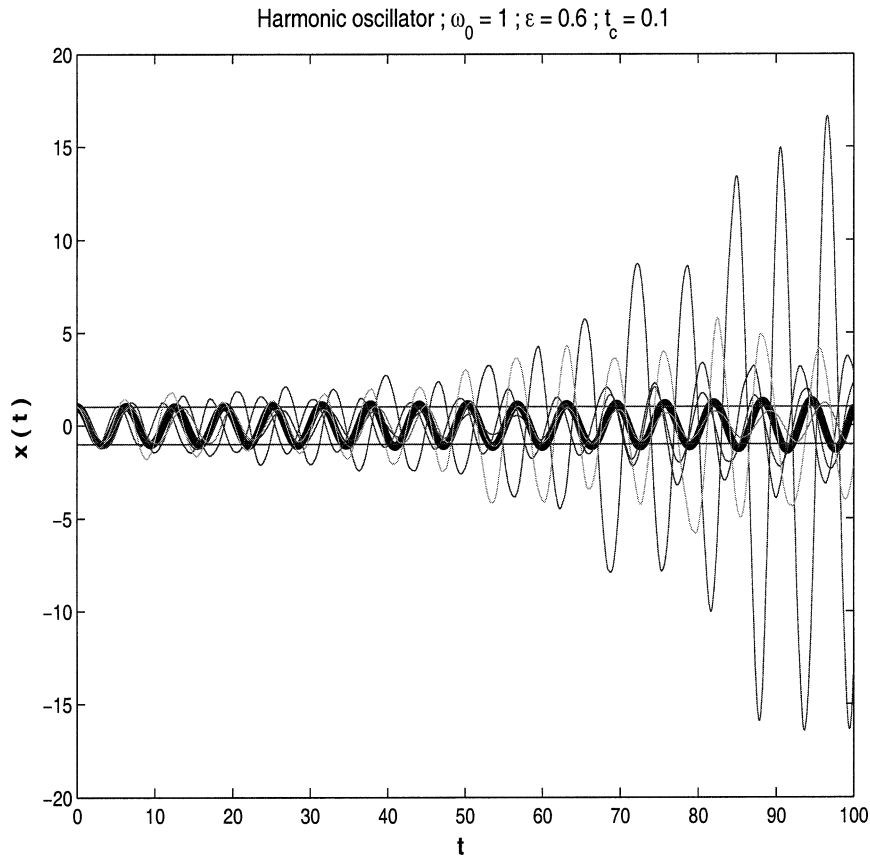
Harmonic oscillator ; $\omega_0 = 1$; $\varepsilon = 0.6$; $t_c = 0.1$

FIG. 8. Ensemble mean displacement as a function of time for the harmonic oscillator with fluctuating spring constant. The initial condition is sure $[x(0) = 1]$. The ensemble mean displacement (thick line) oscillates as in the absence of fluctuations in the spring constant while individual realizations reveal the positive Lyapunov exponent resulting from the fluctuations. Two realizations are shown in the graph (thin lines). The noise is red with autocorrelation time $t_c = 0.1$ and rms amplitude $\epsilon = 0.6$ giving Kubo number $K = t_c\epsilon = 0.06$.

compared to the period of the oscillation. For such short autocorrelation times the position and the fluctuations are effectively uncorrelated, that is, $\langle x(t)\xi(t)\rangle = 0$ as the modulation of the spring constant directly affects the acceleration, which only through integration affects the position of the oscillator. The effect of finite autocorrelation time is shown in Fig. 9. While for short autocorrelation times the ensemble mean displacement follows the ensemble mean equation (66), for finite autocorrelation times the ensemble mean fields decay as can be seen in the middle and bottom panel. It is clear that the stability of the ensemble mean fields is indicative neither of the stability of any single realization nor of the stability of higher moments, all of which are unstable in this example.

8. Conclusions

Developing methods for analyzing perturbation dynamics in uncertain flows is important for advancing understanding of both observations and forecast. Un-

certainty may arise due to ensembling of actual realizations as in observations of Northern Hemisphere winter planetary waves or from a statistical description of a quantity as in a friction parameterization in a forecast model that accounts for variations about the mean dissipation value. In the first example interpreting an observation requires understanding ensemble statistics; in the second interpreting the effect of a statistically described parameter requires related methods.

The concept of moment stability is central to the analysis of uncertain systems and we have used the pdf of growth rates to interpret the relationship among sample and moment growths. Understanding the range of validity of analysis methods is also important and in this work we have identified the Kubo number as a practical measure for the required extent of Gaussianity in modeling uncertain dynamics.

A central concept in GST for uncertain systems is identifying the ensemble and time means through the ergodic assumption. This identification allows application of the powerful methods for ensemble analysis
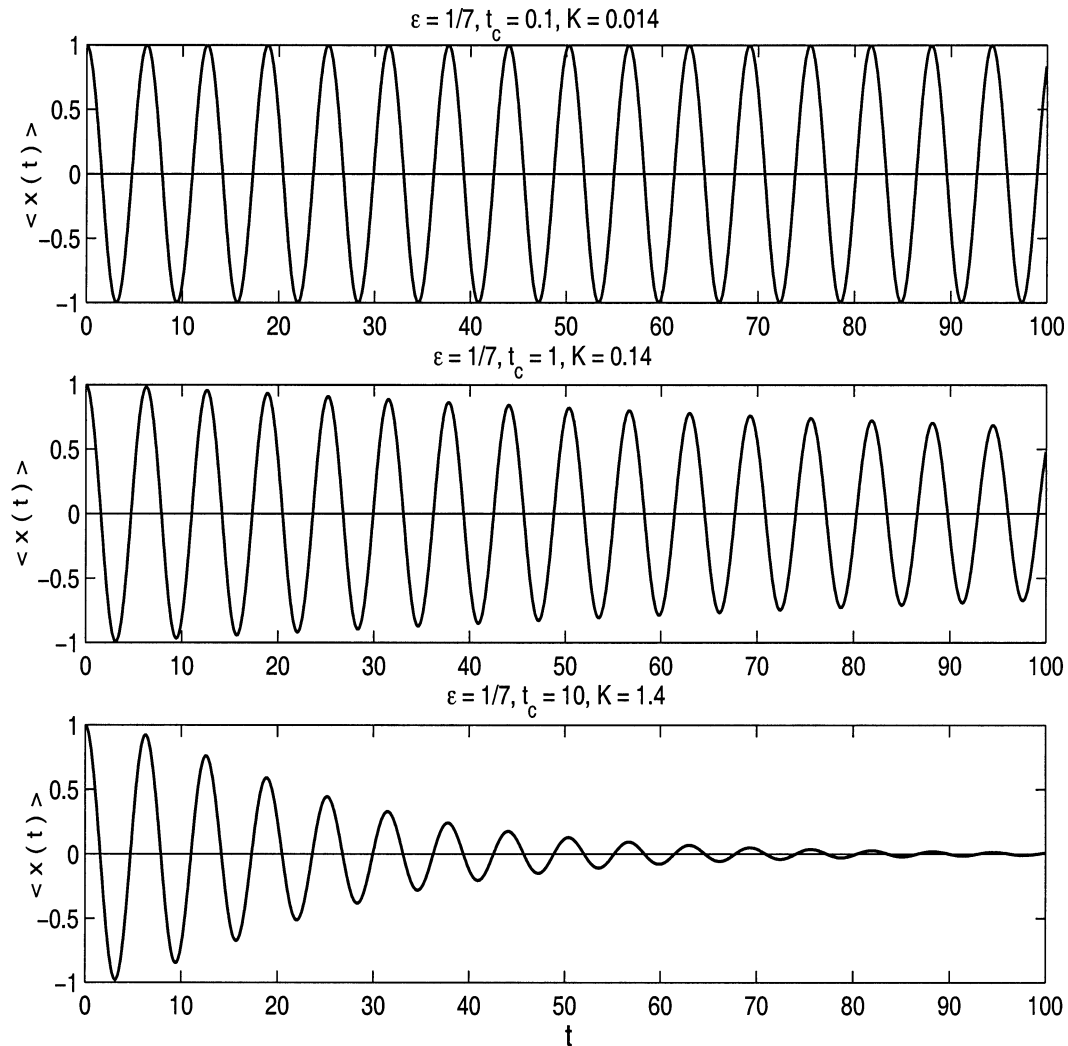
FIG. 9. Ensemble mean displacement as a function of time for the harmonic oscillator with fluctuating spring constant. The rms amplitude of the fluctuation is $\epsilon = 1/7$, and the natural frequency of the oscillator is $\omega = 1$. (top) The ensemble mean displacement for $t_c = 0.1$; in this case there is negligible phase averaging and the solution is as predicted by the short autocorrelation time evolution equation (64), with the ensemble mean oscillating as if there were no fluctuations while the amplitude of individual realizations of the oscillator exponentially diverge. (middle) Intermediate autocorrelation time $t_c = 1$; the effects of correlation between displacement and velocity give a decay of the ensemble mean. (bottom) Long autocorrelation time $t_c = 10$ showing a large effect of phase averaging at this t_c.

to time means of a single realization. Using this assumption a wide class of stationary statistics of a forced uncertain system can be obtained including the frequency response of the ensemble mean as recorded by an averaging instrument.

Optimal excitation plays a central role in GST and in Part I of this paper optimal forcing structures and related optimal response structures have been obtained as a function of forcing frequency for the ensemble mean response as recorded by an averaging instrument.

APPENDIX A

Ensemble Mean Equation for Telegraph Fluctuations

Consider the model system

$$\frac{d\psi}{dt} = \mathbf{A}\psi + \epsilon\xi(t)\mathbf{B}\psi \tag{A1}$$

in which the uncertain operator is assumed to have the form $\mathbf{A}(t, \omega) = \mathbf{A} + \xi(t, \omega)\mathbf{B}$, with $\xi(t, \omega)$ taking only

the values ± 1 and that the jump from one value to the other follows a stationary Markov process known as a telegraph process (Van Kampen 1992), in which the conditional probability that ξ has value ξ_2 at time $t + \Delta t$ given it had the value ξ_1 at time t depends only on the time interval Δt. It is physically reasonable to assume that this conditional probability, $T(\xi_1 | \xi_2; \Delta t)$, for sufficiently small Δt is

$$T(\xi_1 | \xi_2; \Delta t) = \delta_{\xi_1, \xi_2} + w(\xi_1, \xi_2)\Delta t, \quad \text{(A2)}$$

and since

$$\sum_{\xi_2 = \pm 1} T(\xi_1 | \xi_2; \Delta t) = 1, \quad \text{(A3)}$$

transition probability per unit time $w(\xi_1, \xi_2)$ from one state to the other is

$$w(+ | -) = w(- | +) = \frac{\nu}{2}, \quad \text{(A4)}$$

and the probability of state persistence is

$$w(+ | +) = w(- | -) = -\frac{\nu}{2}. \quad \text{(A5)}$$

The resulting two state telegraph process has time-lagged correlation

$$\langle \xi(t + r)\xi(t) \rangle = \exp(-\nu\tau), \quad \text{(A6)}$$

with an autocorrelation time $t_c = 1/\nu$.

The ensemble average of the state for a given initial condition $\psi(0)$ is determined from the ensemble average of the propagator $\mathbf{P}(t)$ by

$$\langle \psi(t) \rangle = \langle \mathbf{P}(t) \rangle \psi(0). \quad \text{(A7)}$$

The propagator $\mathbf{P}(t)$ for a given realization of $\xi(t)$ is

$$\mathbf{P}(t) = \lim_{n \to \infty} \prod_{i=1,n} e^{(\mathbf{A} + \epsilon \xi_i \mathbf{B})\delta t}, \quad \text{(A8)}$$

where $t_i = i\delta t$, and $t_n = t = n\delta t$. The average propagator is the sum of all realizations of the propagator weighted by the probability p that a specific realization $(\xi_1, \ldots, \xi_n)$ of $\xi(t)$ occurs, that is,

$$\langle \mathbf{P}(t) \rangle = \lim_{n \to \infty} \sum_{(\xi_1, \ldots, \xi_n)} p(\xi_1, t_1; \ldots; \xi_n, t_n) \prod_{i=1}^{n} e^{\mathbf{A} + \epsilon \xi_i \mathbf{B}}. \quad \text{(A9)}$$

However, for a Markov process,

$$p(\xi_1, t_1, \ldots; \xi_n, t_n)$$
$$= p(\xi_1, t_1) T(\xi_1 | \xi_2; \delta t) \cdots T(\xi_{n-1} | \xi_n; \delta t). \quad \text{(A10)}$$

Denote as the conditional average, $\langle \mathbf{P}(t) \rangle_\alpha$, the sum of the terms in (A9) for which ξ_n has the well-defined value α at time t. Then $\langle \mathbf{P}(t) \rangle = \langle \mathbf{P}(t) \rangle_+ + \langle \mathbf{P}(t) \rangle_-$. Because ξ is Markov, $\langle \mathbf{P}(t + \delta t) \rangle_\pm$ can be expressed in terms of $\langle \mathbf{P}(t) \rangle_\pm$ as

$$\langle \mathbf{P}(t + \delta t) \rangle_\pm = e^{(\mathbf{A} \pm \epsilon \mathbf{B})\delta t}[T(+ | \pm; \delta t)\langle \mathbf{P}(t) \rangle_+$$
$$+ T(- | \pm; \delta t)\langle \mathbf{P}(t) \rangle_-], \quad \text{(A11)}$$

In the limit of very small δt, $e^{(\mathbf{A} \pm \epsilon \mathbf{B})\delta t}$ can be approximated by $\mathbf{I} + (\mathbf{A} \pm \epsilon \mathbf{B}) \delta t$, and (A2), (A4) and (A5) used to obtain the differential equation governing the evolution of the mean propagator. Because the conditional average propagators $\langle \mathbf{P}(t) \rangle_\pm$ determine the corresponding conditional average of the state $\langle \psi \rangle_\pm$, we immediately obtain the evolution equations for the state averages:

$$\frac{d\langle \psi \rangle_+}{dt} = (\mathbf{A} + \epsilon \mathbf{B})\langle \psi \rangle_+ + \frac{\nu}{2}\langle \psi \rangle_- - \frac{\nu}{2}\langle \psi \rangle_+, \quad \text{(A12)}$$

$$\frac{d\langle \psi \rangle_-}{dt} = (\mathbf{A} - \epsilon \mathbf{B})\langle \psi \rangle_- + \frac{\omega}{2}\langle \psi \rangle_+ - \frac{\nu}{2}\langle \psi \rangle_-, \quad \text{(A13)}$$

from which it follows that the ensemble average of the state $\langle \psi \rangle = \langle \psi \rangle_+ + \langle \psi \rangle_-$ obeys the exact mean equation,

$$\frac{d\langle \psi \rangle}{dt} = \mathbf{A}\langle \psi \rangle + \epsilon^2 \mathbf{B} \int_0^t e^{\mathbf{A}\tau} e^{-\nu\tau} \mathbf{B}\langle \psi(t - \tau) \rangle \, d\tau. \quad \text{(A14)}$$

This development is valid for noncommuting $\mathbf{A}$ and $\mathbf{B}$. The exact equation for the two state telegraphic process, (A14), or variants of it arise also under different noise assumptions, and (A14) can obtained directly from (A1) by perturbation techniques in the limit of small Kubo number, $K \ll 1$ (Bourret 1962; Keller 1962; Brissaud and Frisch 1974; Van Kampen 1974).

APPENDIX B

Remarks on the Ensemble Mean Evolution Equation

In the ensemble mean equation (63) the noise induced correction to the mean operator $\mathbf{A}$ is

$$\mathbf{G}(t) = \epsilon^2 \mathbf{B} \int_0^t ds e^{\mathbf{A}s} \mathbf{B} e^{-\mathbf{A}s} e^{-\nu s}. \quad \text{(B1)}$$

The importance of the noncommutation of $\mathbf{A}$ and $\mathbf{B}$ in shaping this correction is revealed by expanding the integrand in (B1) in a series:

$$e^{\mathbf{A}s} \mathbf{B} e^{-\mathbf{A}s} e^{-\nu s} = e^{-\nu s}\left\{ \mathbf{B} + s(\mathbf{A}, \mathbf{B}) \right.$$
$$\left. + \frac{s^2}{2!}[\mathbf{A}, (\mathbf{A}, \mathbf{B})] + \cdots \right\}, \quad \text{(B2)}$$

where $(\mathbf{A}, \mathbf{B}) \equiv \mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A}$ is the commutator. Using (B2) the equation for the ensemble mean (63) takes the exact form

$$\frac{d\langle \psi \rangle}{dt} = \left(\mathbf{A} + \epsilon^2 \mathbf{B}\left\{ I_0 \mathbf{B} + I_1(\mathbf{A}, \mathbf{B}) \right.\right.$$
$$\left.\left. + \frac{I_2}{2!}[\mathbf{A}, (\mathbf{A}, \mathbf{B})] + \cdots \right\} \right)\langle \psi \rangle, \quad \text{(B3)}$$

where $I_n = \int_0^t s^n e^{-\nu s} \, ds$. This equation is especially useful

if the commutator vanishes at some order so that the series truncates at that order.

It should be noted that for general **A** and **B**, $\mathbf{G}(t)$ may exponentially increase with time. This exponential growth of $\mathbf{G}(t)$ is bounded above by the difference between the decay rates of the most damped and the least damped mode of **A**. Because of the time dependence of $\mathbf{G}(t)$ and the fact that in general it does not commute with **A**, the exponential growth of $\mathbf{G}(t)$ does not necessarily imply the instability of the mean $\langle\psi\rangle$. For example, consider the system

$$\frac{d\psi}{dt} = [\mathbf{A} + \epsilon\eta(t)\mathbf{B}]\psi, \tag{B4}$$

with η a Gaussian noise process with zero mean, unit variance and autocorrelation time $t_c = 1/\nu$, and

$$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -10 \end{pmatrix}, \tag{B5}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{B6}$$

In this case $\mathbf{G}(t)$ grows asymptotically as $e^{(9-\nu)t}$, but for $\epsilon^2/\nu = 1.33$ and $\nu = 1$ Monte Carlo simulation confirms the prediction of (63) that $\langle\psi(t)\rangle$ decays as $e^{-0.86t}$. Note that the white noise limit in this case gives unstable growth at rate 0.33. Examples such as this in which **G** diverges require exponentially small steps for accurate calculation of the propagator governing the ensemble mean evolution.

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