Origin and growth of structures in boundary layer flows

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1 Introduction

A central goal of high Reynolds number fluid dynamics is to gain a comprehensive understanding of the origin and growth of perturbations in shear flows in which the rate of strain of the background velocity provides the source of energy of the perturbations. In all cases the availability of energy for perturbation growth can be determined by linearizing the equations of motion about the appropriate background flow and searching for growing perturbations. If all possible perturbations are examined and only decaying ones are found then it is certain that the background flow will persist when subjected to a sufficiently small disturbance. However, determining the potential for growth of all possible perturbations has not been the historical course of inquiry in stability theory. Rather, traditional stability theory follows the program of Rayleigh (1880) according to which instability is traced to the existence of exponentially growing modes of the linearized dynamic equations. The classical application of the normal mode paradigm envisions unstable modes growing exponentially from infinitesimal beginnings over a large number of e-foldings so that the exponential mode of greatest growth eventually emerges as a finite amplitude wave. This assumption of undisturbed growth is necessary to ensure the asymptotic dominance of the most rapidly growing normal mode which in turn permits the theory to make predictions concerning the structures of finite amplitude. Acceptance of the theory of small oscillations was encouraged by its success in application to problems such as the Rayleigh-Benard and Rayleigh-Taylor problems.

Despite the wide acceptance accorded the normal mode theory there remained difficulties of correspondence in boundary layer shear flows between the observed temporal variation and spatial structure of growing perturbations and the time independent structure of the normal modes. While modal theory predicts that the most unstable perturbations should be 2-dimensional (Squire, 1933) transition proceeds in practice by amplification of fully three-dimensional structures.

Such discrepancies and lingering theoretical questions involving the need to complete the normal modes in the case of linear inviscid dynamics by inclusion of a continuous spectrum of singular neutral modes led to reexamination of the results of Kelvin (1887) and Orr (1907) on the stability of the continuous spectrum by Case (1960). These inquiries showed that the continuous spectrum is stable in the sense that it fails to produce unbounded growth in the limit $t->\infty$ and this negative result was generally interpreted as a proof assuring that the stability of a flow could be determined solely from inspection of its modal spectrum for exponential instabilities. However, it is now more widely appreciated that the modal spectrum only determines stability in the $t -> \infty$ limit and that a more general analysis is necessary to determine the stability properties at finite time. Given that all experiments are conducted in finite time and that the time scale for formation of energetic structures in the boundary layer is rapid finite time stability analysis would seem to be the more appropriate and indeed inquiry shows that in the boundary layer there are non-modal transient disturbances with large growth rates on rapid time scales. The most rapidly growing of these disturbances exhibit transient structural evolution during development that characterizes the observed development of the coherent structures in the boundary layer. Freed of concentrating on the $t -> \infty$ asymptotic, this generalized stability theory allows a much closer correspondence to be made with observed structures which are highly variable both temporally and structurally (Farrell, 1988; Gustavsson, 1991; Butler & Farrell, 1992; Reddy, Schmid & Henningson, 1993; Trefethen *et al.*, 1993; Farrell & Ioannou, 1993a; Reddy & Henningson, 1993; Farrell & Ioannou, 1996).

The existence of a subspace of growing disturbances suggests a mechanism by which the aggregated growth of individual structures supports the statistically steady variance of the fully turbulent flow. It can be shown that the net source of energy to the perturbation field attributable to nonlinear interactions among waves vanishes and it follows that extraction of energy from the forced background flow by the subspace of growing disturbances, which is fully described by linear dynamics, must be responsible for maintaining eddy energy in the fully developed turbulent state (Joseph, 1976 ; Henningson and Reddy, 1994). This observation suggests a mechanistic model for the turbulent state in which the mean flow is subjected to continuous perturbative forcing (Farrell and Ioannou, 1993b, 1994, 1995). The appropriate method of analysis for such a turbulence model is the stochastic dynamics of non-normal linear systems.

The elements of generalized stability theory are reviewed below.

2 Response of the non-normal operator associated with boundary layer flows to impulsive excitation

The equation governing first order perturbation dynamics in the boundary layer is a special case of the general linear dynamical system:

$$\frac{du}{dt} = \mathbf{A}\, u, \tag{1}$$

where $u = [\hat{v}\ , \hat{\eta}]^T$, is the state variable for each streamwise and spanwise Fourier component, $\hat{v}$ is the cross stream perturbation velocity and $\hat{\eta}$ is the cross-stream perturbation vorticity. The dynamical operator in (1) is obtained from the linearized Navier-Stokes equations by eliminating the pressure field using the continuity equation. The operator $\mathbf{A}$ is given by:

$$\mathbf{A} = \begin{bmatrix} \mathbf{L} & 0 \\ \mathbf{C} & \mathbf{S} \end{bmatrix}, \tag{2}$$

with:

$$\mathbf{L} = \Delta^{-1} \left(-i\,k\,U\,\Delta + i\,k\,U'' + \Delta\Delta/R \right), \tag{3.a}$$

$$\mathbf{S} = -i\,k\,U + \Delta/R, \tag{3.b}$$

$$\mathbf{C} = -i\,l\,U', \tag{3.c}$$

and in which the Laplacian operator is given by: $\Delta \equiv \frac{d^2}{dy^2} - K^2$, with K being the total horizontal wavenumber: $K^2 = k^2 + l^2$, and k, l the sreamwise and spanwise wavenumbers respectively. The mean velocity in the streamwise direction which varies only in the cross-stream direction, y, is U and cross-stream derivatives of the mean fields are denoted with a dash. The equations have been rendered non-dimensional with the maximum background velocity in the channel, U_o, and the channel half-width, L so that the Reynolds number is $R \equiv \frac{U_o L}{\nu}$; ν denoting the kinematic viscosity. A well posed inversion of the Laplacian in (3a) requires incorporating the boundary conditions at the channel walls $y = \pm 1$.

The components of the dynamical operator (2) are the Orr-Sommerfeld operator, $\mathbf{L}$, the coupling operator between cross-stream velocity and vorticity, $\mathbf{C}$, which corresponds physically to the generation of cross-stream vorticity by tilting of the mean spanwise vorticity; and the advection-diffusion Squire operator, $\mathbf{S}$.

In the following it is assumed that (2) has been discretized so that $\mathbf{A}$ is the associated linearized dynamical matrix operator. If the background state is steady so that $\mathbf{A}$ is not a function of time then the solution to (1) is explicit:

$$u(t) = e^{\mathbf{A}t}\, u(0) \ . \tag{4}$$

The central distinguishing attribute of $\mathbf{A}$ that determines its transient dynamics is its normality, i.e. whether or not $\mathbf{A}\,\mathbf{A}^\dagger = \mathbf{A}^\dagger\,\mathbf{A}$. If $\mathbf{A}$ commutes with its Hermitian transpose, here indicated by the superscript dagger, then $\mathbf{A}$ is normal and has a complete set of orthogonal eigenvectors. Perturbation growth rate for normal $\mathbf{A}$'s is bounded above by the member of the eigenspectrum of $\mathbf{A}$ with maximum real part.

Because the finite time perturbation dynamics of a non-normal operator can not be ascertained from the spectrum of the operator it is necessary to generalize ideas of perturbation growth by considering the growth, σ, of an arbitrary perturbation u(0) over time t:

$$\sigma^2 = \frac{(u(t)\ ,\ u(t))}{(u(0)\ ,\ u(0))} = \frac{(e^{\mathbf{A}t}u(0)\ ,\ e^{\mathbf{A}t}u(0))}{(u(0), u(0))} = \frac{(e^{\mathbf{A}^\dagger t}\, e^{\mathbf{A}t}u(0)\ ,\ u(0))}{(u(0)\ ,\ u(0))} . \tag{5}$$

The inner product $(\cdot,\cdot)$ generates the Euclidean norm for the vector space: $||\cdot|| = (\cdot,\cdot)^{1/2}$. It follows from (5) that a complete set of orthogonal perturbations $u(0)$ can be ordered in growth over time t by eigenanalysis of the matrix: $e^{\mathbf{A}^\dagger t}\, e^{\mathbf{A}t}$. In particular, the greatest growth over time t as measured by the square of the Euclidean norm is given by the maximum eigenvalue of $e^{\mathbf{A}^\dagger t}\, e^{\mathbf{A}t}$ which is also equal to $||e^{\mathbf{A}t}||^2$ as can be also seen immediately from the singular value decomposition of $e^{\mathbf{A}t}$. The initial condition that gives the maximum growth at a given time is referred to as the optimal perturbation at that time.

There are two asymptotic limits of interest in connection with the excitation of the propagator. In the limit $t -> \infty$ maximum growth is obtained by the eigenfunction associated with the eigenvalue with maximum real part just as normal mode theory would suggest. To see this consider the matrix $\mathbf{E}$ constructed by arranging the eigenvectors of $\mathbf{A}$ as columns in order of growth rate together with the diagonal matrix, Δ, of the associated modal growth factors, from which the following similarity transformation of the propagator can be constructed:

$$e^{\mathbf{A}t} = \mathbf{E}\, e^{\Delta t}\, \mathbf{E}^{-1}\,. \tag{6}$$

In the limit $t -> \infty$ the first column of $\mathbf{E}$ and the first row of $\mathbf{E}^{-1}$ exponentially dominate with amplification factor $e^{\mathrm{Real}(\Delta_{1\ 1})t}$:

$$\lim_{t\ -> \infty} e^{\mathbf{A}t}{}_{\alpha\beta} = \mathbf{E}_{\alpha 1}\ e^{\Delta_{11}t}\ \mathbf{E^{-1}}{}_{1\beta}\,. \tag{7}$$

It can be seen by appeal to Schwartz's inequality that the initial condition of unit norm producing maximum growth over time t is the complex conjugate of $\mathbf{E^{-1}}{}_{1\beta}$ which is the conjugate of the biorthogonal of the leading eigenvector rather than the leading eigenvector itself:

$$\lim_{t\ -> \infty} e^{\mathbf{A}t}{}_{\alpha\beta}(\mathbf{E}^{-1}{}_{1\beta})^* = \mathbf{E}_{\alpha 1}\, e^{\Delta_{11}t}. \tag{8}$$

Modal theory correctly predicts that in the limit $t -> \infty$ that eigenvector which has associated eigenvalue with maximum real part dominates. Not so obvious is the fact that the optimal initial condition with which to excite that mode is the conjugate of the biorthogonal of the dominant mode rather than the mode itself.

Given the observed mean time scale for the formation of coherent structures the $t -> \infty$ asymptotic is not likely to provide a realistic precursor for the formation process. Of greater utility for this purpose is analysis of the $t -> 0$ limit of (5) which controls the initial growth of perturbations. Analysis of this limit provides the maximum possible instantaneous growth rate and the structure that produces this maximum growth rate. The growth rate and the perturbation of maximum instantaneous growth itself also provides a constructive nonlinearly valid bound on the potential for perturbation growth in the flow (Joseph, 1976).

The limit as $t -> 0$ is easily obtained by Taylor expansion of the matrix $e^{\mathbf{A}^\dagger t}\, e^{\mathbf{A}t}$ in (5):

$$\begin{aligned} e^{\mathbf{A}^\dagger t}\ e^{\mathbf{A}t} &\approx (\mathbf{I} + \mathbf{A}^\dagger t)\,(\mathbf{I} + \mathbf{A}t) \\ &= \mathbf{I} + (\mathbf{A} + \mathbf{A}^\dagger)t + \mathbf{0}(t^2), \end{aligned} \tag{9}$$

where **I** is the identity. It follows that a tight upper bound on instantaneous growth rate, and the structure producing this maximum instantaneous growth rate can be found by eigenanalysis of the matrix $\mathbf{A} + \mathbf{A}^\dagger$. The maximum eigenvalue of $(\mathbf{A} + \mathbf{A}^\dagger)/2$ and its associated eigenvector provide the required growth rate and structure. Eigenanalysis of $\mathbf{A} + \mathbf{A}^\dagger$ typically reveals that high growth rates over short times can be realized in boundary layer flows even for low Reynolds numbers for which all normal modes of **A** are damped.

The most relevant time scales for the development of coherent structures in the boundary layer are between the asymptotic limits $t -> 0$ and $t -> \infty$ and for these intermediate time scales the initial and final structures are found most easily from the SVD analysis of the propagator $e^{\mathbf{A}t}$. Given that both asymptotic limits are subsumed it is appropriate to refer to this analysis as the generalized stability analysis of the system (1).

3 Response of non-normal dynamical systems to continuous excitation

Transient growth of disturbances in shear flow can be traced to a substantial subspace of perturbations that extract energy from the background flow. In section 2 analysis of these growing perturbations was framed as an initial value problem involving as a parameter the physically relevant interval in time over which growth occurs. The appropriate method of analysis for studying the maintenance of time mean variance by continuous incoherent forcing is the stochastic dynamics of the associated non-normal system.

The stochastically forced linear dynamical system can be written in the form:

$$\frac{du}{dt} = \mathbf{A}\, u + \mathbf{F}\, \eta(t)\,, \tag{10}$$

in which $\eta(t)$ is a temporally Gaussian white-noise forcing componentwise δ-correlated with zero ensemble mean and unit ensemble covariance:

$$< \eta_i(t_1)\, \eta_j^*(t_2) > \quad = \quad \delta_{\mathrm{ij}}\ \ \delta(t_1 - t_2)\,. \tag{11}$$

The spatial distribution of the forcings is provided by the matrix **F**, and if it is chosen to be unitary the resulting statistics become independent of the particular choice of **F**.

To obtain the stochastic growth of perturbations we first write the forced solution of (10) as:

$$u(t) = \int_0^t e^{\mathbf{A}(t-s)}\, \mathbf{F}\, \eta(s)\, ds\,. \tag{12}$$

The variance maintained by this stochastic forcing is given in the Euclidean norm by:

$$\begin{aligned} < ||u(t)||^2 > &= \left\langle \int_0^t ds \int_0^t ds' \eta^\dagger(s) \mathbf{F}^\dagger e^{\mathbf{A}^\dagger (t-s)} \, e^{\mathbf{A}(t-s')} \mathbf{F} \eta(s') \right\rangle \\ &= \mathrm{Trace}\left(\mathbf{F}^\dagger \int_0^t e^{\mathbf{A}^\dagger (t-s)} \, e^{\mathbf{A}(t-s)} \, ds \;\; \mathbf{F} \right) \\ &= \mathrm{Trace}\left(\mathbf{F}^\dagger \, \mathbf{B} \mathbf{F} \right) \,, \end{aligned} \tag{13}$$

revealing that the hermitian operator

$$\mathbf{B}^t = \int_0^t e^{\mathbf{A}^\dagger s} \, e^{\mathbf{A} s} \, ds \,, \tag{14}$$

accumulates the perturbation growth when all perturbations are stochastically excited. This operator should be compared with the operator $e^{\mathbf{A}^\dagger t} \, e^{\mathbf{A} t}$ eigenanalysis of which reveals the optimal perturbation growth as we have seen in the previous section. An alternative and computationally preferable method for calculating the stochastic dynamical operator $\mathbf{B}^t$ results from differentiating (14) with respect to time to obtain:

$$\frac{d\mathbf{B}^t}{dt} = \mathbf{I} + \mathbf{A}^\dagger \, \mathbf{B}^t + \mathbf{B}^t \, \mathbf{A} \,, \tag{15}$$

in which $\mathbf{I}$ is the identity matrix.

In direct analogy with the analysis of optimal growth in the previous section a complete set of orthogonal forcings forming the columns of a unitary $\mathbf{F}$ can be found for the stochastic variance at time t in (13) by eigenanalysis of the positive definite hermitian $\mathbf{B}^t$. If the operator $\mathbf{A}$ is asymptotically stable a stationary solution is obtained in which the eigenfunctions of $\mathbf{B}^\infty$ are ordered according to their contribution to the variance of the statistically steady state. The forcings ordered in this way will be referred to as stochastic optimals.

The stochastic optimals most effectively excite the stationary variance and should be contrasted with the orthogonal structures that most effectively span the maintained variance, which are commonly referred to as the EOF's of the dynamical system. The stochastic optimals bear a relationship to the EOF's in the stochastic analysis analogous to that between the optimal excitation and the optimal response in the SVD analysis of the propagator of the initial value problem. To obtain the EOF's we need first to form the correlation matrix:

$$\begin{aligned} \mathbf{C}^t_{ij} &= < u_i(t) \, u^*_j(t) > \\ &= \left(\int_0^t e^{\mathbf{A}(t-s)} \, \mathbf{F}\mathbf{F}^\dagger \, e^{\mathbf{A}^\dagger (t-s)} \, ds \right)_{\mathrm{ij}} \,, \end{aligned} \tag{16}$$

which satisfies:

$$\frac{d\mathbf{C}^t}{dt} = \mathbf{F}\mathbf{F}^\dagger + \mathbf{A} \, \mathbf{C}^t + \mathbf{C}^t \, \mathbf{A}^\dagger \,. \tag{17}$$

Each eigenvalue of the positive definite hermitian operator $\mathbf{C}^t$ equals the variance accounted for, under unbiased forcing and at time t, by the pattern of its corresponding eigenvector, and the pattern that corresponds to the largest eigenvalue contributes most to the perturbation variance at t.

If $\mathbf{A}$ is normal and the forcing unitary ($\mathbf{F}\mathbf{F}^\dagger = \mathbf{I}$) then $\mathbf{A}$, $\mathbf{B}^t$, $\mathbf{C}^t$ commute and the stochastic optimals, the EOF's, and the modes of the dynamical system coincide. For such a system eigenanalysis of $\mathbf{A}$ suffices for understanding the statistics of the perturbations in the linear limit. In contrast, for non-normal systems the stochastic optimals, the EOF's and the modes of the dynamical operator are all distinct (Farrell & Ioannou, 1993b; 1995).

If $\mathbf{A}$ is asymptotically stable the system approaches a statistically steady state as t increases in which $\mathbf{B}^\infty$ and $\mathbf{C}^\infty$ satisfy the Lyapunov equations :

$$\begin{aligned} \mathbf{A}\mathbf{C}^\infty + \mathbf{C}^\infty\mathbf{A}^\dagger &= -\,\mathbf{F}\mathbf{F}^\dagger \\ \mathbf{A}^\dagger\,\mathbf{B}^\infty + \mathbf{B}^\infty\,\mathbf{A} &= -\,\mathbf{I}\,. \end{aligned} \tag{18}$$

The Lyapunov equations (18) are readily solved for $\mathbf{B}^\infty$ and $\mathbf{C}^\infty$ given the asymptotically stable operator $\mathbf{A}$ and the forcing correlation matrix $\mathbf{F}$.

4 Conclusions

The turbulent state of wall bounded pipe and channel flows is characterized by energetic interactions between the highly sheared mean flow of the boundary layer and a subset of coherent disturbances having the form of streamwise streaks and associated streamwise vortices. Because of the very high shears found in the boundary layer, perturbation dynamics may plausibly be anticipated to be dominated by the mean straining field which forms the primary interaction between the mean flow and the perturbations and which is incorporated in the linear dynamical operator. This fundamental linearity of boundary layer turbulence is demonstrated by comparisons performed between simulations with and without inclusion of nonlinear wave interactions (Lee *et al.*, 1990). Despite this evident simplicity of the dynamics, straightforward calculation of the eigenspectrum of the linearized dynamical operator fails to produce structures with the observed form of streamwise vortices. This failure of correspondence can be understood from the perspective of analysis of the non-normal operator associated with the linear dynamics as due to the fact that while the perturbations of maximal growth in the linear problem take the form of streamwise rolls these are not the eigenfunctions of the linearized operator, rather they are the optimal structures identified with the first singular vectors of the propagator arising from the dynamical operator in an appropriate norm and with an appropriate time interval for development.

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