Comments on "Small Error Dynamics and the Predictability of Atmospheric Flows"

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In his article, Farrell (1990) discusses some general aspects of error growth in the atmosphere and illustrates them on a baroclinic shear flow and on the barotropic vorticity equation. The basic step of his development is to associate error growth with deviations from a "basic" state describing an idealized atmosphere free of the well-known variability of real-world atmospheric flows and to study the response of this state to small initial fluctuations. It is argued that the eigenvalues of the linearized operator associated to this state will govern the dynamics of the error growth in the atmosphere on short time scales.

Classical theory of cyclogenesis initiated by Charney (1947) proposes to look at the real-world atmosphere as a result of the instability of an idealized basic state. We regard this inspiring view, which has led to many fruitful developments, as an efficient way to model atmospheric complexity. On the other hand, we believe that identifying the deviation from this idealized and unrepresentative state as the "error" and associating it with the "predictability" of the atmosphere may lead to a number of ambiguities. Indeed, recent years have witnessed major developments in the field of nonlinear dynamical systems, as a result of which error growth has acquired a specific significance that calls for a reappraisal of the concept of error growth as used by Farrell. Specifically, one now knows that large classes of dynamical systems exhibit deterministic chaos. In such a regime the long-time solutions of the evolution laws lying on the system's stable attractor display aperiodicity and unpredictability in both space and time. This is reflected by sensitivity with respect to initial perturbations on the attractor, while perturbations transversal to the attractor are damped (Guckenheimer and Holmes 1983). In other words, even though the basic state is now a stable solution of the evolution laws (notice that such a state displays a complex time dependence), small perturbations are nevertheless amplified, and *it is this phenomenon that is referred to as error growth* in dynamical systems theory (Lorenz 1984). The mechanism of amplification turns out to depend on the system's parameters and is, therefore, an *intrinsic property* of the dynamics. In contrast, as we show in the sequel in Farrell's formulation, error growth depends on the basic-state guess and follows no systematic trend. We do not question at this point the linearization procedure, which is legitimate for short time behavior, but rather, the use of a basic state of the type considered by Farrell.

Suppose first that the basic state is not an exact solution of the full nonlinear equations. Under these conditions, the deviation from this state will give rise to a transient that will eventually die out as the system settles to its final solution. In our view, there is no special reason why this transient, which, in addition, depends on the particular approximation chosen, will be related at all to predictability. In particular, the fact that the eigenvalues of the linear operator **L** depend on this approximation introduces an undesirable subjective element and identifies error to our ignorance rather than to the system's behavior as described by the model adopted.

An additional deficiency is that by considering a reference state that is not an exact solution of the full nonlinear equations, one is no longer entitled to use a linear *homogeneous* equation for the error field. Indeed, let [cf., Farrell's Eq. (2.1)]

$$\frac{d\psi}{dt} = \mathbf{F}(\psi) \tag{1}$$

be the initial equation. Setting $\psi = \psi_0 + \delta\psi$, one obtains to the leading order

$$\frac{d\delta\psi}{dt} = \mathbf{F}(\psi_0) + \mathbf{L}\delta\psi \tag{2}$$

since $\mathbf{F}(\psi_0) \neq 0$. To see the possible role of the inhomogeneous term $\mathbf{F}(\psi_0)$, consider the one-variable equation

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$$\frac{dx}{dt} = a - \lambda x, \qquad (3)$$

whose solution is

$$x(t) = \frac{a}{\lambda} + \left(x_0 - \frac{a}{\lambda}\right)e^{-\lambda t}, \qquad (4)$$

where x_0 is the initial error.

Suppose that $\lambda > 0$ (operator L has negative eigenvalue). One would be tempted to conclude that the "error" x is then damped. This is not always the case, however: From Eq. (4), we see that x will actually *grow* (nonexponentially) if $x_0 < a/\lambda$. This shows the ambiguities that may be born if error is measured from the "wrong" reference state (here $x = 0$) rather than from the "right" one [here $x = a/\lambda$, the exact steady-state solution of (3)]. Obviously, when many variables are present, the solution will be even more complicated and the "error" x can behave in many possible ways, none of which will be indicative of predictability.

Suppose now that in addressing error growth and predictability one has access to a time-independent reference state ψ_0 that is an exact steady-state solution of Eq. (1). Two cases may arise.

i) *ψ_0 is stable:* after an initial transient, small deviations are damped or remain bounded. The question of error growth and predictability simply does not arise unless one arbitrarily defines the initial transient as such. This is contrary to elementary ideas and practice of differential equations and dynamical systems theory since the type of transient that will be realized depends entirely on the initial perturbation and follows no systematic trend.

ii) *ψ_0 is unstable:* initial deviations are amplified, the system leaves ψ_0, and, after a transient period, it reaches a stable (generally time-dependent) solution of Eq. (1). Now there is no reason to believe that a physically reasonable system will find itself spontaneously on an unstable situation such as ψ_0. Once again, error growth and predictability seem to be linked to the inadequacy of the unrepresentative guessed solution ψ_0 rather than to the intrinsic properties of the system itself. It might, of course, happen that ψ_0 provides a reasonable approximation of the exact, *time-dependent* stable solution for very short times. This practical expedient should, however, not be confused with a fundamental theory of predictability.

In conclusion, we suggest that:

i) The concept of error growth around a stable time-independent or even a (multi-) periodic solution makes no sense. Transient behavior, possibly accompanied with accidental temporary deviations, may occur but depends on the details of initial perturbations and bears no systematic relation with the attractor properties.

ii) It is highly unlikely that the instability of the atmosphere will be associated with the passage from an unstable (and consequently unphysical) solution. The occurrence of an unstable dynamics on a stable attractor constitutes an appealing alternative. Since the only known type of attractor displaying these properties is a chaotic one, error growth is therefore to be formulated around such a basic state. Typically, such a state has a very complex dependence in space and time and is, for this reason, available only numerically. In this perspective, error growth in the atmosphere bears (in an average sense) a clear-cut relation with the attractor properties.

We end this comment by pointing out that error growth need not necessarily be exponential even in chaotic systems. As shown recently on both simple mathematical models (Nicolis and Nicolis 1991) and models of interest in atmospheric physics (Nicolis 1992), exponential growth may hold true in chaotic dynamical systems whose attractors are everywhere hyperbolic. On the other hand, in multifractal attractors and large classes of *spatially extended* systems, error growth is likely to be nonexponential (Nicolis et al. 1992; Trevisan 1992).

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